Write Kirchhoff-integral for each component of $\underline{E}(\underline{x})$:

$\boxed{\text{VECTORIAL THEORY II}}$ – for scatterers of general form.

$$\underline{E}(\underline{x}) = \oint_S \left\{ \underline{E}(\hat{\underline{n}}' \cdot \underline{\nabla}' G) - G(\hat{\underline{n}}' \cdot \underline{\nabla}')\underline{E} \right\} da'$$

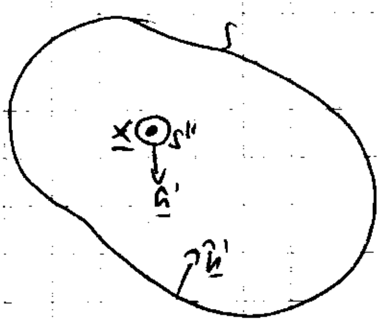
Problem: mixup of $\underline{E}$-components at bounds.

later: free-space G.F.

$$G(\underline{x}, \underline{x}') = \frac{e^{ikR}}{4\pi R}, \quad \underline{R} = \underline{x} - \underline{x}'$$

$$\underline{\nabla}' G(\underline{x}, \underline{x}') = -\frac{e^{ikR}}{4\pi R}\left[ik - \frac{1}{R}\right]\frac{\underline{R}}{R}$$

To avoid singularities in V: add surface S'' around $\underline{x}$, then



$$\oint_{S+S''} (\text{same}) = 0, \quad \oint_{S''} (\text{same}) = -\underline{E}(\underline{x})$$

benefit: no singularities in V

$$-G(\hat{\underline{n}}' \cdot \underline{\nabla}')\underline{E} = -(\hat{\underline{n}}' \cdot \underline{\nabla}')G\underline{E} + \underline{E}(\hat{\underline{n}}' \cdot \underline{\nabla}' G) \quad \Rightarrow$$

$$0 = \oint_{S+S''} \left\{ 2\underline{E}(\hat{\underline{n}}' \cdot \underline{\nabla}' G) - \underbrace{(\hat{\underline{n}}' \cdot \underline{\nabla}')G\underline{E}}_{\text{e.g. for x-component, this is}} \right\} da'$$

Divergence theorem

$$\int -\hat{\underline{n}}' \cdot \underline{\nabla}' G E_x \, da' = \int -\underline{\nabla}' G E_x \cdot d\underline{a}' = \int \nabla'^2 G E_x \, d^3x' \ \checkmark$$

Process: $$\int \underline{\nabla}'^2 G\underline{E}\, d^3x = \int \underline{\nabla}'(\underline{\nabla}' \cdot G\underline{E})\, d^3x - \int \underline{\nabla}' \times (\underline{\nabla}' \times G\underline{E})\, d^3x$$

$$= -\int (\underline{\nabla}' \cdot G\underline{E})\, d\underline{a}' + \int \hat{\underline{n}}' \times (\underline{\nabla}' \times G\underline{E})\, da'$$

(divergence theorem) (other divergence theorem)

So: $$0 = \oint_{S+S''} \left[2\underline{E}(\hat{\underline{n}}' \cdot \underline{\nabla}' G) - \hat{\underline{n}}'(\underline{\nabla}' \cdot G\underline{E}) + \hat{\underline{n}}' \times \underline{\nabla}' \times G\underline{E} \right] da'$$

$\underline{\text{Use}}$: $\underline{\nabla}' \cdot G\underline{E} = \underline{E}\cdot\underline{\nabla}'G + \cancel{G\underline{\nabla}'\cdot\underline{E}} = \underline{E}\cdot\underline{\nabla}'G$

$\underline{\text{and}}$: $\underline{n}'\times(\underline{\nabla}'\times G\underline{E}') = \underline{n}'\times\Big[G\,\underbrace{\underline{\nabla}'\times\underline{E}}_{\text{F.L.}=i\omega\underline{B}}\Big] + \underline{n}'\times\big[\underline{\nabla}'G\times\underline{E}\big]$ ⊛

$$0 = \oint\Big[2\underline{E}(\underline{n}'\cdot\underline{\nabla}'G) - (\underline{E}\cdot\underline{\nabla}'G)\underline{n}' + \underline{n}'\times[\underline{\nabla}'G\times\underline{E}] + G\,i\omega\,\underline{n}'\times\underline{B}\Big]da'$$

$= \underline{E}(\underline{n}'\cdot\underline{\nabla}'G) + \big[\underline{E}(\underline{n}'\cdot\underline{\nabla}'G) - (\underline{E}\cdot\underline{\nabla}'G)\underline{n}'\big]$

$= \underline{E}(\underline{n}'\cdot\underline{\nabla}'G) + \cancel{\underline{\nabla}'G\times[\underline{E}\times\underline{n}']}$

$= \cancel{\underline{E}(\underline{n}'\cdot\underline{\nabla}'G)} + \underbrace{(\underline{n}'\times\underline{E})\times\underline{\nabla}'G}_{\text{keep}}$

$= \underbrace{(\underline{n}'\cdot\underline{E})\underline{\nabla}'G}_{\text{keep}} - \cancel{(\underline{n}'\cdot\underline{\nabla}'G)\underline{E}}$

$$0 = \oint_{\boxed{S+S''}}\Big[i\omega(\underline{n}'\times\underline{B})G + (\underline{n}'\times\underline{E})\times\underline{\nabla}'G + (\underline{n}'\cdot\underline{E})\underline{\nabla}'G\Big]da' = -E(x) + \oint_{\boxed{\text{Sandy}}} \text{Eq. 10.89}$$

⌈can do same analysis with $c\underline{B}$, but use at location ⊛:

M.A.L $c(\underline{\nabla}\times\underline{B}) = c(-i\omega)\varepsilon_0\mu_0\underline{E} = -i\frac{\omega}{c}\underline{E}$ in place of $i\omega\underline{B}$

$\Rightarrow$ replacements: $\underline{E}\to c\underline{B}$ and $\underline{B}\to -\frac{\underline{E}}{c}$ ⌋

$$\boxed{\begin{aligned}\underline{E}(\underline{x}) &= \oint_S\Big[i\omega[\hat{\underline{n}}'\times\underline{B}]G + (\hat{\underline{n}}'\times\underline{E})\times\underline{\nabla}'G + (\hat{\underline{n}}'\cdot\underline{E})\underline{\nabla}'G\Big]da' \\ c\underline{B}(\underline{x}) &= \oint_S\Big[-ik[\hat{\underline{n}}'\times\underline{E}]G + (\underline{n}'\times c\underline{B})\times\underline{\nabla}'G + (\underline{n}'\cdot c\underline{B})\underline{\nabla}'G\Big]da'\end{aligned}}$$

(mistake in 10.90)

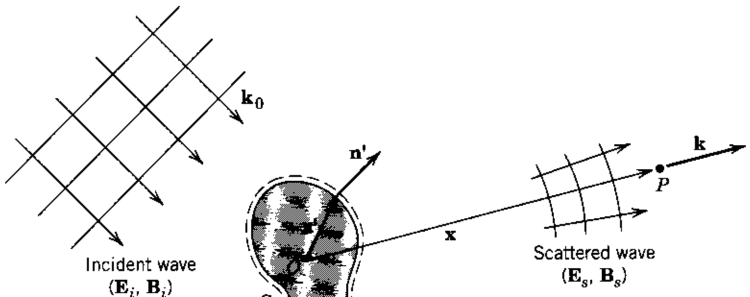
$$\oint_{S_2} \to O\left(\frac{1}{r_0}\right)$$

and the contribution from the integral over S_2 vanishes as $r_0 \to \infty$. For the geometry of Fig. 10.7, then, with S_2 at infinity, the electric field in region II satisfies the *vector Kirchhoff integral relation*,

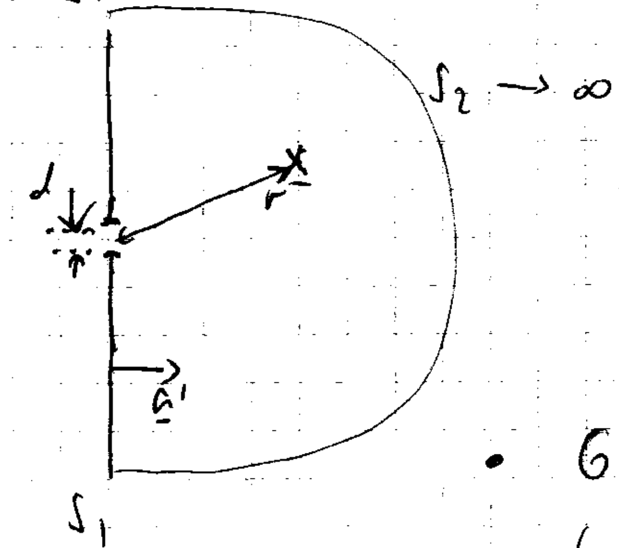
$$\mathbf{E}(\mathbf{x}) = \oint_{S_1} [i\overset{\omega}{\cancel{k}}(\mathbf{n}' \times \mathbf{B})G + (\mathbf{n}' \times \mathbf{E}) \times \nabla' G + (\mathbf{n}' \cdot \mathbf{E})\nabla' G]\, da' \qquad (10.90)$$

where G is given by (10.76) and the integral is only over the finite surface S_1.

It is useful to specialize (10.90) to a scattering situation and to exhibit a formal expression for the scattering amplitude as an integral of the scattered fields over S_1. The geometry is shown in Fig. 10.9. On both sides of (10.90) the



Consider:



- Free-space G.F.
- can show $\oint_{S_2} \to 0$
- use far-field expression

- $G = e^{ikr} \frac{1}{4\pi r} e^{-ik\hat{\underline{x}}\cdot\underline{x}'} = \frac{e^{ikr}}{4\pi r} e^{-i\underline{k}\cdot\underline{x}'}$

(page 484: replace $e^{-ik\underline{n}'\cdot\underline{x}}$ with $e^{-i\underline{k}\cdot\underline{x}'}$)

- $\underline{\nabla}' G = -i\underline{k}\, G$

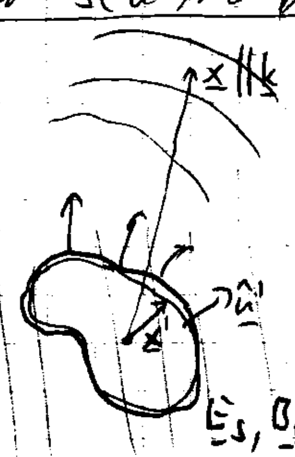
[notations: 10.1 - 10.4: $\hat{n}_0$ = direction of incident wave

$\underline{n}$ = " of exit wave (observation direction)]

now: $\hat{n}'$ = surface normal, $\underline{k}$ = direction of observation $\times k$

$\underline{k}_0$ = incident direction $\times k$

Insert scattered fields on surface into integral relations for fields:



(imply observer doesn't see $\underline{E}_{in}$)

then:

$\underline{E}_s = \underline{E}_{total} - \underline{E}_{in}$

$\Downarrow$ Far-field integral relations for $\underline{E}_s$, $\underline{B}_s$

$$\underline{E}_s(\underline{x}) = \frac{e^{ikr}}{4\pi r} \oint_S \left\{ i\omega\left[\underline{n}' \times \underline{B}_s(\underline{x}')\right] + i\underline{k} \times \left[\underline{n}' \times \underline{E}_s(\underline{x}')\right] - i\underline{k}\left[\underline{n}' \cdot \underline{E}_s(\underline{x}')\right] \right\} e^{-i\underline{k}\cdot\underline{x}'} da'$$

$$c\underline{B}_s(\underline{x}) = \frac{e^{ikr}}{4\pi r} \oint_S \left\{ -ik\left[\underline{n}' \times \underline{E}_s(\underline{x}')\right] + i\underline{k} \times \left[\underline{n}' \times c\underline{B}_s(\underline{x}')\right] - i\underline{k}\left[\underline{n}' \cdot c\underline{B}_s(\underline{x}')\right] \right\} e^{-i\underline{k}\cdot\underline{x}'} da'$$

Definition: scattering amplitude

$$\underline{E}_s(\underline{x}) = \frac{e^{ikr}}{r} \underline{F}(\underline{k}, \underline{k}_0)$$

$$\boxed{\underline{F}(\underline{k}, \underline{k}_0) = \frac{i}{4\pi} \oint e^{-i\underline{k}\cdot\underline{x}'} \left\{ \omega[\underline{n}' \times \underline{B}_s] + \underline{k} \times [\underline{n}' \times \underline{E}_s] - \underline{k}[\underline{n}' \cdot \underline{E}_s] \right\} da' \quad 10.91} \quad (*)$$

Field must be transverse to $\underline{k}$ (in far-field) => can write as $\underline{k}\times \dots$

Do: $\omega \underline{n}' \times \underline{B} - \underline{k}[\underline{n}' \cdot \underline{E}]$ $\qquad \hat{\underline{k}} \cdot \underline{E} = 0$

$$= \omega \underline{n}' \times \underline{B} - k\left[\hat{\underline{k}} \cdot (\underline{n}' \cdot \underline{E}_s)\hat{\underline{k}} - \hat{\underline{k}} \cdot (\underline{n}' \cdot \hat{\underline{k}})\underline{E}\right]$$

F.L. $\nabla \times \underline{E} = i\omega \underline{B}$

far field $i\underline{k} \times \underline{E} = i\omega \underline{B}$

$$\boxed{\underline{k} \times \underline{E} = \omega \underline{B}}$$

$$= \omega(\underline{n}' \times \underline{B}) - \underline{k}\left\{\hat{\underline{k}} \cdot \left[\underline{n}' \times (\hat{\underline{k}} \times \underline{E})\right]\right\}$$

$$= \omega(\underline{n}' \times \underline{B}) - \underline{k}\left[\hat{\underline{k}} \cdot \left(\underline{n}' \times \frac{\omega}{k} \underline{B}\right)\right]$$

$$= \omega\left[\underline{n}' \times \underline{B} - \hat{\underline{k}}\left(\hat{\underline{k}} \cdot (\underline{n}' \times \underline{B})\right)\right]$$

$$= -\hat{\underline{k}} \times \hat{\underline{k}} \times [\underline{n}' \times \omega \underline{B}] = -\frac{c}{k} \underline{k} \times \underline{k} \times [\underline{n}' \times \underline{B}] \quad \Big|\ \omega = ck$$

$$\underline{F}(\underline{k}, \underline{k}_0) = \frac{i}{4\pi} \oint_S e^{-i\underline{k}\cdot\underline{x}'} \left\{ -\frac{c}{k} \underline{k} \times \underline{k} \times (\underline{n}' \times \underline{B}_s) + \underline{k} \times (\underline{n}' \times \underline{E}_s) \right\} da'$$

$$\boxed{\underline{F}(\underline{k}, \underline{k}_0) = \frac{-i}{4\pi} \underline{k} \times \oint_S e^{-i\underline{k}\cdot\underline{x}'} \left[\frac{c\,\underline{k} \times (\underline{n}' \times \underline{B}_s)}{k} - \underline{n}' \times \underline{E}_s \right] da'} \quad 10.92$$

$\cancel{\underline{E}_s = \underline{E}_{total} - \underline{E}_{in}}$ $\qquad$ ~~$\underline{k}_0$, $\underline{E}_{in}$, props implicit in $\underline{E}_s$, $\underline{B}_s$~~

- polarization analysis via $\underline{\varepsilon}^* \cdot \underline{F}$, start from (*)
- $\frac{dP}{d\Omega} = \frac{1}{2Z_0} r^2 \underline{E} \cdot \underline{E}^* = \frac{1}{2Z_0} \left|\underline{F}(\underline{k}, \underline{k}_0)\right|^2$; $\frac{d\sigma}{d\Omega} = \frac{dP}{d\Omega} \Big/ I_{inc}$

$I_{inc} = \frac{1}{2Z_0} \underline{E}_{in} \underline{E}_{in}^*$

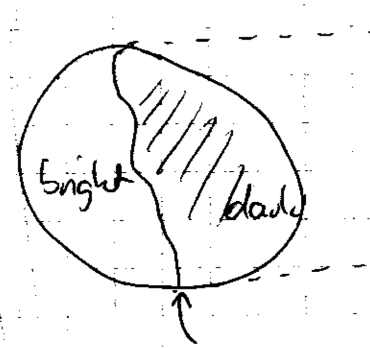
Scattering in short-wavelength limit, Apps: radar, lidar etc. (38)

Use $\underline{E}_s = \frac{e^{ikr}}{r}\, \underline{F}(\underline{k}, \underline{k}_0)$

$$\underline{\varepsilon}^* \cdot \underline{F}(\underline{k}, \underline{k}_0) = \frac{i}{4\pi} \int e^{-i\underline{k}\cdot\underline{x}'} \left[\omega\, \underline{\varepsilon}^* \cdot (\underline{n}' \times \underline{B}_s) + \underline{\varepsilon}^* \cdot (\underline{k} \times (\underline{n}' \times \underline{E}_s)) \right] da' \qquad (*)$$

(to see, do $\underline{\varepsilon}^* \cdot$ Eq 10.91)

Specify $\underline{E}_s, \underline{B}_s$:



dark side: $\underline{E} = \underline{E}_s + \underline{E}_{in} = 0$

$\underline{E}_s = -\underline{E}_{in}$

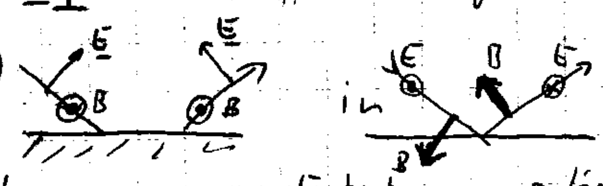
$\underline{B}_s = -\underline{B}_{in}$

$\lambda \ll R$ => well defined "shadow line" (important)

illuminated side (for conducting scatter)

- in (*) only need tangential fields
- $\underline{E}_{tan} = \underline{E}_{in,tan} + \underline{E}_{sc,tan} = 0$ $\quad \underline{E}_{sc,tan} = -\underline{E}_{in,tan}$
- $\underline{B}_\perp = 0$ afforded by $\quad \underline{B}_{sc,tan} = +\underline{B}_{in,tan}$

[(think of it locally)



$\underline{E}$ in plane satisfies $B_\perp = 0$, $\underline{E}_{tan} = 0$

$\underline{E} \perp$ plane satisfies B.C. $B_\perp = 0$, $\underline{E}_{tan} = 0$]

- Calculate contributions from shadow and light regions separately. Shadow region: Insert the following into Eq. 10.93

$\underline{E}_s = -\underline{E}_{in} = -\underline{\varepsilon}_0 E_0 e^{i\underline{k}_0\cdot\underline{x}}$, $\underline{B} = -\frac{1}{kc}\, \underline{k}_0 \times \underline{E}_{in} = -\frac{E_0}{kc} e^{i\underline{k}_0\cdot\underline{x}}\, \underline{k}_0 \times \underline{\varepsilon}_0$

$$\epsilon^* \cdot E_{sh} = \frac{E_0}{4\pi i}\int_{\text{shadow}} \epsilon^* \cdot \Big[\underbrace{\overset{\text{from } B}{n' \times (k_0 \times \epsilon_0)} + \overset{\text{from } E}{k \times (\hat{n}' \times \epsilon_0)}}\Big]\, e^{i(k_0 - k)\cdot x'}\, da'$$

$$= k_0 \times (\hat{n}' \times \epsilon_0) + \left[(\hat{n}' \cdot \epsilon_0) k_0 - \cancel{(k_0 \cdot \epsilon_0)\hat{n}'}\right]$$

$$= \frac{E_0}{4\pi i}\int_{sh} \epsilon^* \cdot \left[(k + k_0) \times (n' \times \epsilon_0) + (\hat{n}' \cdot \epsilon_0)\, k_0 \right] e^{i(k_0 - k)\cdot x'}\, da'$$

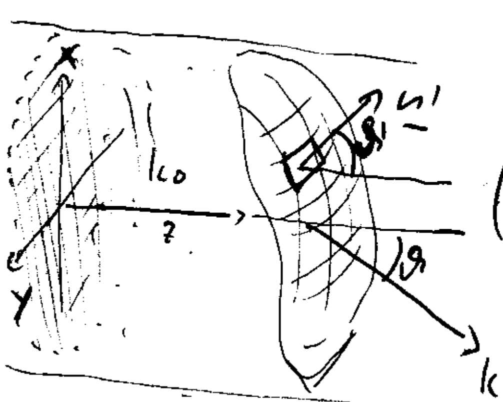
<u>dominant contributions</u> if $k \approx k_0$, b/c then exponential oscillates less. (note here that $kd >> 1$). For $k \simeq k_0$, it is $\epsilon^* \cdot k_0 \approx 0$ => 2nd term negligible

$\approx$ $\epsilon^* \cdot k_0 \simeq k \sin\theta$

$$\epsilon^* \cdot E_{sh} \approx \frac{E_0}{4\pi i}\, \epsilon^* \cdot \int 2 k_0 \times (\hat{n}' \times \epsilon_0)\, e^{i(k_0 - k)\cdot x'}\, da'$$

$$= \frac{E_0}{2\pi i}\, \epsilon^* \cdot \int \left[\cancel{(k_0 \cdot \epsilon_0) n'} - (k_0 \cdot n')\epsilon_0\right] e^{i(k_0 - k)\cdot x'}\, da'$$

$$= \frac{i E_0}{2\pi} (\epsilon^* \cdot \epsilon_0) \int_{sh} e^{i(k_0 - k)\cdot x'}\, (k_0 \cdot n')\, da'$$



$$k_0 \cdot n'\, da' = k\, da' \cos\theta' = k\, dx\, dy = k\, d^2 x_\perp$$

("area projection law")

$$(k_0 - k)\cdot x' = \underbrace{(1 - \cos\theta) k}_{\approx 0} - k_x x - k_y y \approx - k_\perp \cdot x_\perp$$

$$\boxed{\Rightarrow \epsilon^* \cdot E_{sh} \approx \frac{i E_0 k}{2\pi} (\epsilon^* \cdot \epsilon_0) \int_{\text{shadow}} e^{-i k_\perp \cdot x_\perp}\, d^2 x_\perp}$$

Note: $\underline{\varepsilon}^* \cdot \underline{F}_{sh} = \frac{ikE_0}{2\pi} (\underline{\varepsilon}^* \cdot \underline{\varepsilon}_0) \underbrace{\int_{\text{shadow}} e^{-i\underline{k}_\perp \cdot \underline{x}_\perp} d^2 x_\perp}$

$F(x,y) = \begin{cases} 1, (x,y) \text{ inside profile} \\ 0, (x,y) \text{ outside profile} \end{cases}$ | 2D FT of projection F of scatterer outline on plane $\perp$ to k_0

so: only outline of scatterer matters, properties of 3D shape irrelevant.

- stealth designs useless for forward scattering.
- recall $\underline{E}_{sc} = \frac{e^{ikr}}{r} \underline{F}(\underline{k}, \underline{k}_0)$

Scattering from illuminated regions

conductor only. Also, recall only need $\underline{E}_{tan}$, $\underline{B}_{tan}$

$$\underline{E}_{s,tan} = -\underline{E}_{in,tan}$$
$$\underline{B}_{s,tan} = + \underline{B}_{in,tan}$$

absence of $\ominus$ is only difference to shadow region

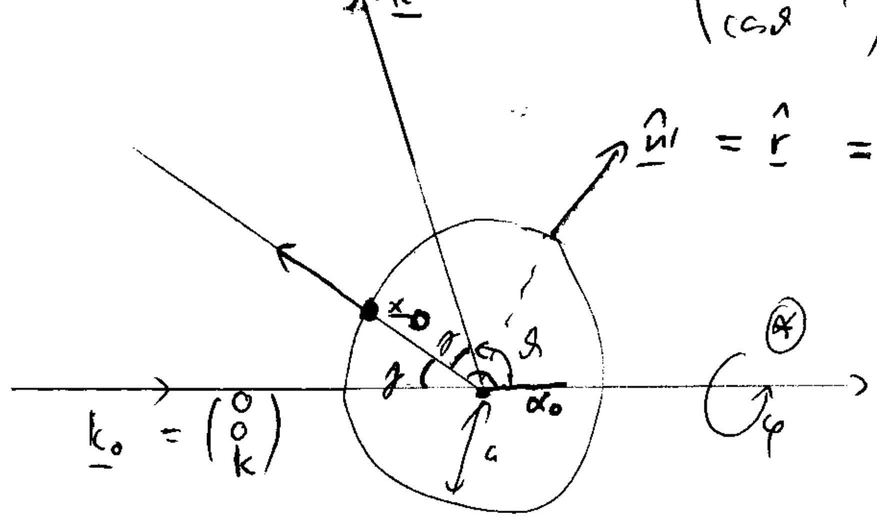
so: take previous intermediate result and flip the signs of $\underline{k}_0$ – terms (which were all associated with a $\underline{B}_s$)

$$\underline{\varepsilon}^* \cdot \underline{F}_{ill} = \frac{E_0}{4\pi i} \int_{ill.} \underline{\varepsilon}^* \cdot \underbrace{[(\underline{k} - \underline{k}_0) \times (\hat{n}' \times \underline{\varepsilon}_0) - (\hat{n}' \cdot \underline{\varepsilon}_0)\underline{k}_0]}_{\text{can't neglect anything}} \underbrace{e^{i(\underline{k}_0 - \underline{k}) \cdot \underline{x}'}}_{\text{oscillates rapidly}} da'$$

numerics, or do $\longrightarrow$ $\boxed{\text{seek stationary-phase condition.}}$

$$\underline{k} = k\,r = k\begin{pmatrix}\sin\vartheta\cos\varphi\\ \sin\vartheta\sin\varphi\\ \cos\vartheta\end{pmatrix}$$

$$\underline{\hat{n}}' = \underline{\hat{r}} = \begin{pmatrix}\sin\alpha\cos\beta\\ \sin\alpha\sin\beta\\ \cos\alpha\end{pmatrix} = \frac{\underline{x}_0}{a}$$



$$\gamma = \tfrac{1}{2}(\pi - \vartheta)$$
$$\alpha_0 = \pi - \gamma = \frac{\pi}{2} + \frac{\vartheta}{2}$$

$\underline{x}_0$ = point on surface which leads to specular reflection from direction $\underline{k}_0$ into direction $\underline{k}$.

<u>Phase</u>: $(\underline{k}_0 - \underline{k})\cdot\underline{x}' = ka\,[-\sin\vartheta\cos\varphi\sin\alpha\cos\beta - \sin\vartheta\sin\varphi\sin\alpha\sin\beta$
insert above vectors $+ (1-\cos\vartheta)\cos\alpha]$

$$\Phi(\alpha,\beta) = ka\left[(1-\cos\vartheta)\cos\alpha - \sin\vartheta\sin\alpha\cos(\beta-\varphi)\right]$$

<u>Phase is stationary for $\underline{x}' = \underline{x}_0$</u>; $\underline{x}_0 = a\,\underline{\hat{n}}'(\alpha_0\beta_0)$

to find α_0, β_0: $\dfrac{\partial\Phi}{\partial\alpha} = \dfrac{\partial\Phi}{\partial\beta} = 0 \;\Rightarrow\; \boxed{\begin{aligned}\alpha_0 &= \frac{\pi}{2}+\frac{\vartheta}{2}\\ \beta_0 &= \varphi\end{aligned}}$ note (*)

(obvious) angles of specular reflection.

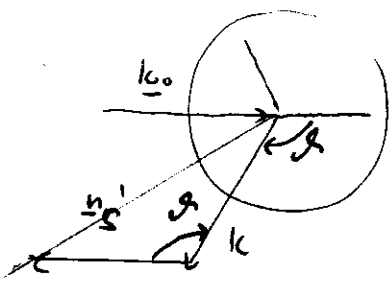
<u>Expand phase</u> around α_0, β_0. With $x = \alpha - \alpha_0$
$y = \beta - \beta_0$

$$\Phi(x,y) = c_1 + c_2 x^2 + c_3 y^2 = \dots$$
$$= -2ka\sin\frac{\vartheta}{2}\left[1 - \frac{1}{2}\left(x^2 + \cos^2\frac{\vartheta}{2}\,y^2\right) + \dots\right]$$

$$\varepsilon^* \cdot E_{ill} = \frac{E_0}{4\pi i}\int_{ill} \varepsilon^* \cdot \left[(k-k_0)\times(n_s'\times\varepsilon_0) - (n_s'\cdot\varepsilon_0)k_0\right] e^{i\Phi(x,y)}\, da'$$

use *fixed* value of stationary-phase point.

Noting



$$\boxed{\hat n_s' = \frac{k-k_0}{|k-k_0|}} \quad \text{with } |k-k_0| = k\sqrt{2-2\cos\vartheta}$$

law of cosines

$$da' = a^2 \sin_\alpha d\alpha\, d\beta \approx a^2 \sin\alpha_0\, d\alpha\, d\beta = \sqrt{\frac{\cos\vartheta+1}{2}}\, a^2\, d\alpha\, d\beta$$

$$= a^2\, d\alpha\, d\beta \underbrace{\sin\left(\frac{\pi}{2}+\frac{\vartheta}{2}\right)}_{=\cos\frac{\vartheta}{2} = \sqrt{\frac{\cos\vartheta+1}{2}}}$$

Also: $\varepsilon^*\cdot\left[|k-k_0|\,\hat n_r\times(\hat n_r\times\varepsilon_0) - (n_r\cdot\varepsilon_0)k_0\right]$

can add because $\varepsilon^*\cdot k = 0$

$$= \varepsilon^*\cdot|k-k_0|\left[(\hat n_r\cdot\varepsilon_0)\hat n_r - \varepsilon_0 \underbrace{- \hat n_r\cdot\varepsilon_0\frac{k_0}{|k-k_0|} + \frac{k}{|k-k_0|}}_{=(\hat n_r\cdot\varepsilon_0)\hat n_r}\right]$$

$$= |k-k_0|\,\varepsilon^*\cdot\left(-\varepsilon_0 + 2\hat n_r(\hat n_r\cdot\varepsilon_0)\right)$$

$$= \sqrt{2-2\cos\vartheta}\, k\, \varepsilon^*\cdot\underline{\varepsilon_r}$$ polarization of reflected light

---

$$\varepsilon^*\cdot E_{ill} = \frac{E_0}{4\pi i}\underbrace{\sqrt{\frac{\cos\vartheta+1}{2}}\, k\sqrt{2-2\cos\vartheta}}_{k\sin\vartheta}\, a^2\int e^{i\Phi(x,y)}\,dx\,dy$$

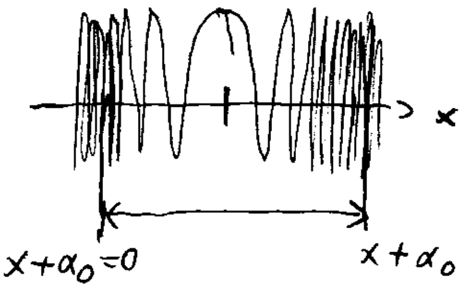
$$= \frac{E_0 a^2 k}{4\pi i}\sin\vartheta\, e^{-i2ka\sin\frac{\vartheta}{2}}\,\varepsilon^*\cdot\varepsilon_r\int e^{ika\sin\frac{\vartheta}{2}x^2}dx\int e^{ika\sin\frac{\vartheta}{2}y^2}dx$$

(doesn't factorize so nicely for elliptical surface)

(43)

With $\int_{-\infty}^{\infty} e^{ika\sin\frac{\vartheta}{2} x^2} dx = \sqrt{\frac{\pi i}{ka\sin\frac{\vartheta}{2}}} = \int \dots dy$

(can integrate to ∞ b/c $ka \gg 1$)



b/c of fast wiggling can just as well integrate to ∞

(and exclude non-illuminated regions)

$$\underline{\varepsilon}^* \cdot \underline{E}_{ill} = \frac{E_0 a^2 k}{4\pi i} \sin\vartheta\, e^{-i2ka\sin\frac{\vartheta}{2}} \underline{\varepsilon}^* \cdot \underline{\varepsilon}_r \frac{\pi i}{ka\sin\frac{\vartheta}{2}}$$

$$\frac{\sin\vartheta}{\sin\frac{\vartheta}{2}} = 2\frac{\sin\vartheta/2 \cos\vartheta/2}{\sin\vartheta/2}$$

$$\boxed{\underline{\varepsilon}^* \cdot \underline{E}_{ill} = \frac{E_0 a}{2} e^{-i2ka\sin\frac{\vartheta}{2}} \underline{\varepsilon}^* \cdot \underline{\varepsilon}_r \cos\frac{\vartheta}{2}}$$

phase shift of specular beam

polarization $\underline{\varepsilon}_r = -\underline{\varepsilon}_0 + 2\hat{\underline{n}}_r(\hat{\underline{n}}_r \cdot \underline{\varepsilon}_0)$

note on Polarization

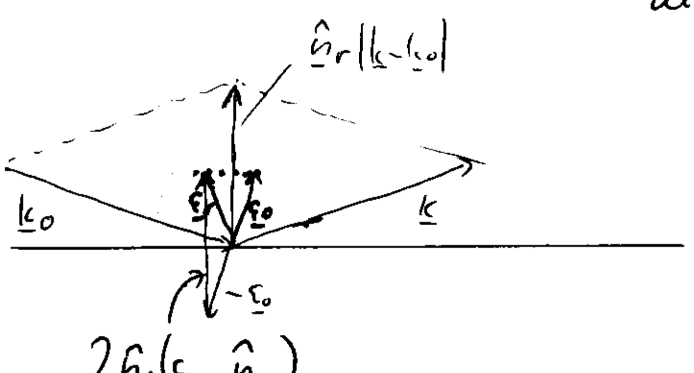


diagram: for $\underline{\varepsilon}_0$ in-plane

$\underline{\varepsilon}_r = -\underline{\varepsilon}_0 + 2\hat{\underline{n}}_r(\hat{\underline{n}}_r \cdot \underline{\varepsilon}_0)$

for $\underline{\varepsilon}_0 \perp$ plane

$\underline{\varepsilon}_r = -\underline{\varepsilon}_0 = -\underline{\varepsilon}_0 + 2\hat{\underline{n}}_r \underbrace{(\hat{\underline{n}}_r \cdot \underline{\varepsilon}_0)}_{=0 \text{ for } \varepsilon_0 \perp \text{ plane}}$

to satisfy $\underline{E}_{tan} = 0$

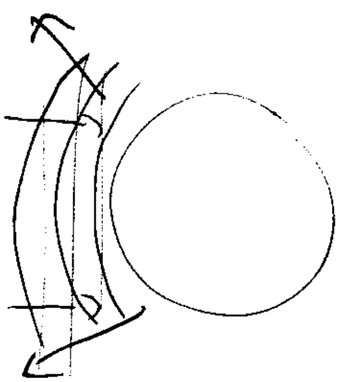
$\boxed{\text{so: in any case, } \underline{\varepsilon}_r = -\underline{\varepsilon}_0 + 2\hat{\underline{n}}_r(\hat{\underline{n}}_r \cdot \underline{\varepsilon}_0) \text{ correct.}}$

Note on trends:

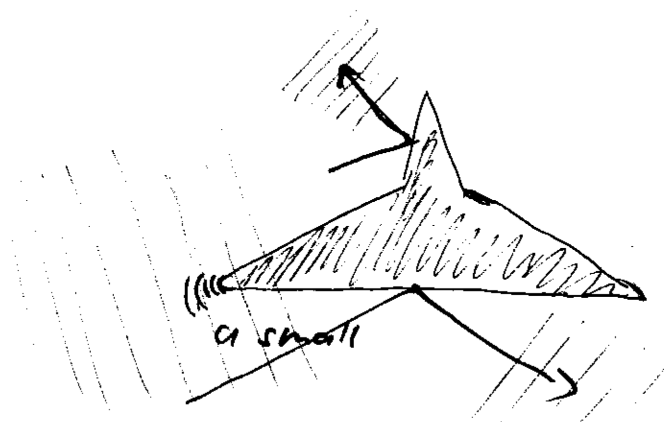
$$\left| \underline{s}^{*} \cdot \underline{E}_{i\|} \right|^2 \propto a^2$$

better "stealth design" in aircraft if all corners have small radii of curvature.

normal:



stealth



explain how to "hide" something at optical frequencies...

"Anti-stealth" radar

- forward scattering
- long wavelength

Optical theorem (proof only)

total field on scatterer surface: $\underline{E} = \underline{E}_{sc} + \underline{E}_{in}$

scattered field: $\underline{E}_{sc}$

incident field: $\underline{E}_{in}$

power absorbed: $$P_{abs} = -\frac{1}{2\mu_0} \int \mathrm{Re}\, \hat{\underline{n}}' \cdot (\underline{E} \times \underline{B}^*)\, da'$$

surface of scatterer S

scattered power: $$P_{scat} = +\frac{1}{2\mu_0} \int_S \mathrm{Re}\, \hat{\underline{n}}' \cdot (\underline{E}_{sc} \times \underline{B}_{sc}^*)\, da'$$

power removed from incident beam: $P_{total} = P_{abs} + P_{scat}$

insert
$\underline{E} = \underline{E}_{in} + \underline{E}_{sc}$
$\underline{B} = \underline{B}_{in} + \underline{B}_{sc}$

$$= -\frac{1}{2\mu_0} \int_S \mathrm{Re}\Big[\hat{\underline{n}}' \cdot \big(\underline{E}_{in} \times \underline{B}_{sc}^* + \underline{E}_{sc} \times \underline{B}_{in}^* + \underbrace{\underline{E}_{in} \times \underline{B}_{in}}_{\text{integrates to zero}}\big)\Big]\, da'$$

$$= -\frac{1}{2\mu_0} \int_S \mathrm{Re}\Big[\hat{\underline{n}}' \cdot \big(\underline{E}_{in}^* \times \underline{B}_{sc} + \underline{E}_{sc} \times \underline{B}_{in}^*\big)\Big]$$

| b/c $\hat{\underline{n}}'$ real

Insert $\underline{E}_{in} = E_0\, \underline{\varepsilon}_0\, e^{i \underline{k}_0 \cdot \underline{x}'}$, $c\underline{B}_{in} = \frac{1}{k} \underline{k}_0 \times \underline{E}_{in}$

$$P_{total} = -\frac{1}{2\mu_0} \int_S \mathrm{Re}\Big[E_0^*\, e^{-i\underline{k}_0 \cdot \underline{x}'}\, \hat{\underline{n}}' \cdot \Big\{\underline{\varepsilon}_0^* \times \underline{B}_{sc} + \underline{E}_{sc} \times \Big(\frac{\underline{k}_0}{kc} \times \underline{\varepsilon}_0^*\Big)\Big\}\Big]\, da'$$

$$= -\frac{1}{2\mu_0} \int_S \mathrm{Re}\Big[E_0^*\, e^{-i\underline{k}_0 \cdot \underline{x}'} \Big\{\underline{\varepsilon}_0^* \cdot [\underline{B}_{sc} \times \hat{\underline{n}}'] + \Big(\frac{\underline{k}_0}{kc} \times \underline{\varepsilon}_0^*\Big) \cdot [\hat{\underline{n}}' \times \underline{E}_{sc}]\Big\}\Big]\, da'$$

$$= -\frac{1}{2\mu_0} \int_S \mathrm{Re}\Big[E_0^*\, e^{-i\underline{k}_0 \cdot \underline{x}'} \Big\{\underline{\varepsilon}_0^* \cdot [\underline{B}_{sc} \times \hat{\underline{n}}'] + \underline{\varepsilon}_0^* \cdot \Big([\hat{\underline{n}}' \times \underline{E}_{sc}] \times \frac{\underline{k}_0}{kc}\Big)\Big\}\Big]\, da'$$

$$= \frac{1}{2\mu_0} \int_S \mathrm{Re}\left[E_0^* e^{-i\underline{k}_0\cdot\underline{x}'} \underline{\varepsilon}_0^* \cdot \left\{ \hat{n}' \times \underline{B}_{sc} + \frac{\underline{k}_0}{kc} \times (\hat{n}' \times \underline{E}_{sc}) \right\} \right] da'$$

We compare the total (absorbed + scattered) power with the forward scattering amplitude at the incident polarization, i.e. set $\underline{k} = \underline{k}_0$ and $\underline{\varepsilon} = \underline{\varepsilon}_0$

$$\underline{\varepsilon}_0^* \cdot \underline{F}(\underline{k}_0, \underline{k}_0) = \frac{i}{4\pi} \int_S e^{-i\underline{k}_0\cdot\underline{x}'} \underline{\varepsilon}_0^* \cdot \left\{ kc\, \hat{n}' \times \underline{B}_{sc} + \underline{k}_0 \times (\hat{n}' \times \underline{E}_{sc}) \right\} da'$$

to see that $\quad \mathrm{Im}\left[\underline{E}_0^* \underline{\varepsilon}_0^* \cdot \underline{F}(\underline{k}_0, \underline{k}_0) \right] = P_{total} \frac{2\mu_0 kc}{4\pi} = P_{total} \frac{Z_0 k}{2\pi}$

$$P_{total} = \frac{2\pi}{k Z_0} \mathrm{Im}\left[E_0^* \underline{\varepsilon}_0^* \cdot \underline{F}(\underline{k}_0, \underline{k}_0) \right]$$

Defining the normalized scattering amplitude

$$\boxed{f(\underline{k}, \underline{k}_0) := \frac{\underline{F}(\underline{k}, \underline{k}_0)}{E_0}}$$

and recalling that the total scattering cross section is "power / incident intensity"

$$\sigma_t = \frac{P_{total}}{I_{inc}} = \frac{P_{total}}{\frac{1}{2Z_0} E_0 E_0^*}$$

we find

$$\frac{\sigma_t}{2Z_0} E_0 E_0^* = \frac{2\pi}{k Z_0} \mathrm{Im}\left[E_0 E_0^* \underline{\varepsilon}_0^* \cdot f(\underline{k}_0, \underline{k}_0) \right]$$

$$\boxed{\sigma_t = \frac{4\pi}{k} \mathrm{Im}\left[\underline{\varepsilon}_0^* \cdot f(\underline{k}_0, \underline{k}_0) \right]}$$

This is the optical theorem, which relates total (absorption + scattering) cross section with the normalized forward scattering amplitude (both for polarization $\underline{\varepsilon}_0$). Recall the scattered electric field is $\underline{E}_s = \frac{e^{ikr}}{r} \underline{F}(\underline{k}, \underline{k}_0) = \frac{e^{ikr}}{r} E_0 f(\underline{k}, \underline{k}_0)$.
The theorem is closely related with dielectric properties of ensembles of scatterers ($\Rightarrow$ name)