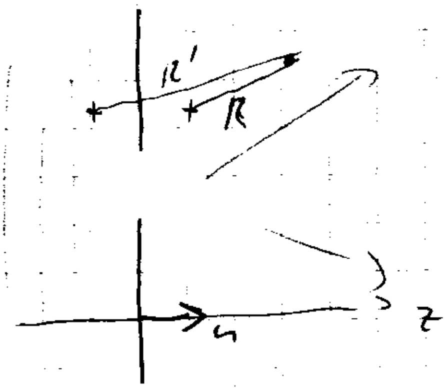
Vectorial theory I – plane at $z=0$, $\sigma=\infty$



- microscopic theory
- planar charges and currents
- Symmetry ⟹

$$\underline{E} = \underline{E}^{(0)} + \underline{E}'$$
$$\underline{B} = \underline{B}^{(0)} + \underline{B}'$$

fields due to currents in screen

E_x', E_y', B_z' even in z
E_z', B_x', B_y' odd in z

vector potential with Neumann GF:

$$\underline{A}'(\underline{x}) = -\int G_N \frac{\partial \underline{A}'}{\partial z'}\, da' \qquad G_N = \frac{1}{4\pi}\left(\frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'}\right)$$

sources only on $z=0$, where $R=R'$

$$= -\frac{1}{2\pi}\int \frac{e^{ikR}}{R}\begin{pmatrix}\partial_z A_x' \\ \partial_z A_y' \\ \partial_z A_z'\end{pmatrix} da'$$

$A_z \equiv 0$ b/c $\partial_z = 0$

$$-\frac{1}{2\pi}\int \frac{e^{ikR}}{R}\begin{pmatrix}\partial_z A_x' - \partial_x A_z' \\ -\partial_y A_z' + \partial_z A_y' \\ 0\end{pmatrix} da' = -\frac{1}{2\pi}\int \frac{e^{ikR}}{R}\begin{pmatrix} +B_y' \\ -B_x' \\ 0\end{pmatrix} da'$$

$$= -\frac{1}{2\pi}\int \frac{e^{ikR}}{R}\left(-\hat{\underline{z}} \times \underline{B}'\right) da' = \frac{1}{2\pi}\int \frac{e^{ikR}}{R}\, \hat{\underline{n}} \times \underline{B}'\, da' \quad \text{good.}$$

Since B_x', B_y' odd ⟹ $\int$ over screen without holes only

$B_x = B_y = 0$ in holes

$$\boxed{\underline{B}'(\underline{x}) = \frac{1}{2\pi}\, \underline{\nabla} \times \int_{\text{screen} \setminus \text{holes}} \left(\underline{n} \times \underline{B}'\right) \frac{e^{ikR}}{R}\, da'}$$

Symmetry of source-free equations in $\underline{E}$, $\underline{B}$

$$\Rightarrow \quad \underline{E}'(\underline{x}) = \pm \frac{1}{2\pi} \underline{\nabla} \times \int_{\text{screen + holes}} (\hat{n} \times \underline{E}') \frac{e^{ikR}}{R} da' \qquad \begin{matrix} z>0 \\ z<0 \end{matrix}$$

[sign b/c $\underline{E}' \times \underline{B}' \simeq \hat{z}$ for $z>0$

$\underline{E}' \times \underline{B}' \simeq -\hat{z}$ for $z<0$ (direction of flow)]

Note: $\underline{n} \times \underline{E}' \neq 0$ on screen, b/c $\underline{E}'$ is only the scattered field

but: tangential field of total, $\underline{E} = \underline{E}' + \underline{E}^{(0)}$, vanishes on screen

thus:

$$\underline{E}'(\underline{x}) = \pm \frac{1}{2\pi} \underline{\nabla} \times \int_{\text{holes only}} \hat{n} \times \underbrace{(\underline{E}' + \underline{E}^{(0)})}_{\underline{E}} \frac{e^{ikR}}{R} da'$$

$$\mp \frac{1}{2\pi} \underline{\nabla} \times \int \hat{n} \times \underline{E}^{(0)} \ldots$$

Define:

$$\underline{E}_{diff}(\underline{x}) = \frac{1}{2\pi} \underline{\nabla} \times \int_{\text{holes}} \hat{n} \times \underline{E} \frac{e^{ikR}}{R} da'$$

total field

total tangential field

$z>0$: $\underline{E} = \underline{E}_{diff}(\underline{x}) = \underline{E}' + \underline{E}^{(0)}$

$z<0$: $\underline{E} = \underline{E}^{(0)} - \underline{E}_{diff} + \underline{E}^{(1)}$

field reflected from hole-less screen

Smythe-Kirchhoff integral

- plane screen, $\sigma = 0$, at $z=0$
- near-field o.k.

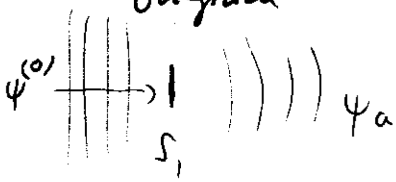
Babinet's principle

for scalar theory

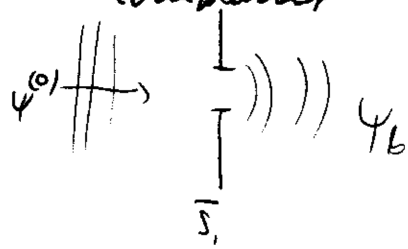
Kirchhoff integral:

$$\psi(\underline{x}) = -\frac{1}{4\pi}\int_{S_1} \frac{e^{ikR}}{R}\,\underline{n}'\cdot\left[\underline{\nabla}'\psi + ik\left(1+\frac{i}{kR}\right)\frac{\underline{R}}{R}\psi\right] da' \quad ; \quad \underline{R} = \underline{x}-\underline{x}'$$

original



complement



$$\psi_a+\psi_b = -\frac{1}{4\pi}\int_{\text{all surface}} \dots \, da' = \psi^{(0)}$$

(Huygens' principle for full plane)

$$\boxed{\psi_a+\psi_b = \psi^{(0)}}$$ exact.

screen and its complement have same diffraction pattern.

Vectorial Babinet principle

for thin, perfectly well conducting screen at $z=0$

① original problem

$\underline{E}^{(0)}, \underline{B}^{(0)}, \underset{\text{screen}}{S}, \underset{\text{scattered fields}}{\underline{E}, \underline{B}}$

complementary problem

② $\underline{E}_c^{(0)} = c\,\underline{B}^{(0)}$,

$\underline{B}_c^{(0)} = -\frac{1}{c}\underline{E}^{(0)}$

screen $\bar{S}$, $\underline{E}_c, \underline{B}_c$

Smythe-Kirchhoff-Integral

①: get $\underline{E}(\underline{x}) = \frac{1}{2\pi}\underline{\nabla}\times\int_{\text{apert } S}(\underline{n}\times\underline{E})\frac{e^{ikR}}{R}\,da'$ $\qquad \underline{R} = \underline{x}-\underline{x}'$

(total field $\underline{E} = \underline{E}' + \underline{E}^{(0)}$)

②: get $\underline{B}_c'(\underline{x}) = \frac{1}{2\pi}\nabla\times\int_{\text{surface of } \bar{S} = \text{aps of } S}(\underline{n}\times\underline{B}_c')\frac{e^{ikR}}{R}\,da'$ $\qquad$ (Eq. 10.97)

with $\underline{B}_c' + \underline{B}_c^{(0)} = \underline{B}_c$ $\qquad$ (10.94) $\qquad$ ($\underline{B}_c'$ = scattered field only)

①, ② same integrals $\Longrightarrow$

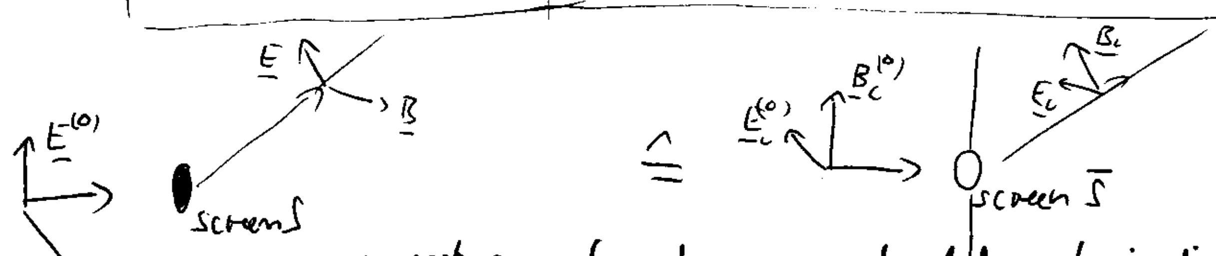
vector Babinet

$\underline{E}(\underline{x}) = c\,\underline{B}_c'$ and $\underline{B}_c' = \underline{B}_c - \underline{B}^{(0)}$

$$\boxed{\begin{aligned}\underline{E}(\underline{x}) - c\underline{B}_c &= -c\,\underline{B}_c^{(0)} = \underline{E}^{(0)}\\ \underline{B}(\underline{x}) + \underline{E}_c/c &= \underline{B}^{(0)}\end{aligned}}$$

scalar Babinet

$\psi_a + \psi_b = \psi$



must complement screen and rotate polarization by $90°$
(doesn't matter for unpolarized light)

Fraunhofer diffraction (far-field diffraction)

let $r \to \infty$ in Eq. 10.79

$$G = \frac{e^{ikR}}{4\pi R} \approx \frac{e^{ikr}}{4\pi r} e^{-i\underline{k}\cdot\underline{x}'}$$

$$\underline{\nabla}' G \simeq -i\underline{k}\, G$$

note: $\underline{k}$ = direction of outgoing wave $\cdot k$, $\underline{k} = k\frac{\underline{x}}{r}$

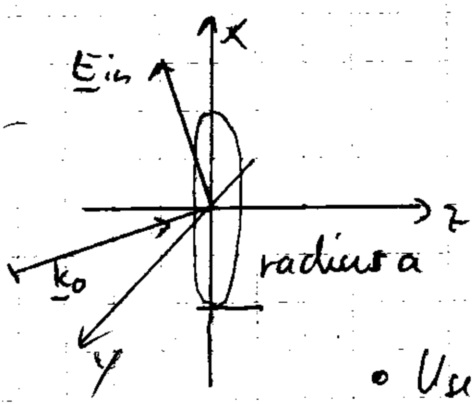
In that limit, the Smythe-Kirchhoff integral is

$$\boxed{\underline{E}(\underline{x}) = \frac{i e^{ikr}}{2\pi r}\underline{k} \times \int_{S_1} \hat{\underline{n}} \times \underline{E}(\underline{x}')\, e^{-i\underline{k}\cdot\underline{x}'}\, da'}$$

far field

$\hat{\underline{n}}$ = screen normal

Calculation for circular hole:



$$\underline{E}_{in} = E_0(\hat{\underline{x}}\cos\alpha - \hat{\underline{z}}\sin\alpha)\, e^{i\underline{k}_0\cdot\underline{x}}$$

with $\underline{k}_0 = k(\hat{\underline{x}}\sin\alpha + \hat{\underline{z}}\cos\alpha)$

- Use $\underline{E}_{in}$ on rhs of Smythe-Kirchhoff formula
- $\int$ over aperture

Result: $\underline{E}(\underline{x}) = \frac{i e^{ikr}}{r} a^2 E_0 \cos\alpha\, (\underline{k}\times\hat{\underline{y}})\, \frac{J_1(ka\xi)}{ka\xi}$

where $\xi = \sin^2\vartheta + \sin^2\alpha - 2\sin\vartheta\sin\alpha\cos\varphi$

$\underline{k}$ = direction of observation $\times k$ $= k\frac{\underline{x}}{r}$

(ϑ, φ) are coordinates of $\underline{k}$

From $\underline{E}(\underline{x})$, obtain $\frac{dP}{d\Omega} = r^2 \frac{1}{2Z_0}\underline{E}\cdot\underline{E}^*$, $\frac{d\sigma}{d\Omega} = \frac{dP}{d\Omega}\Big/\left[\frac{1}{2Z_0}E_0 E_0^*\right]$

and $\sigma = \int_{\varphi=0}^{2\pi}\int_{\vartheta=0}^{\pi/2} \frac{d\sigma}{d\Omega}\, d\Omega$ (note that integration goes only from $\vartheta = 0$ to $\vartheta = \frac{\pi}{2}$)

Transmission factor $T = \frac{\sigma}{a^2\pi}$

$$T \to \begin{cases} 1, & ka \gg 1 \;✓ \\ \frac{1}{3}(ka)^2 \cos\alpha \to 0, & ka \ll 1 \;✓ \end{cases}$$

Comparison with scalar theory

$$\boxed{\psi(\underline{x}) = -\frac{e^{ikr}}{4\pi r}\int_{\text{holes}} e^{-i\underline{k}\cdot\underline{x}'}\left[\hat{\underline{n}}\cdot\underline{\nabla}'\psi(\underline{x}') + i\underline{k}\cdot\hat{\underline{n}}\,\psi(\underline{x}')\right]da'}$$

$\underline{k} = k\frac{\underline{x}}{r}$, $\hat{\underline{n}}$ = surface normal

Kirchhoff integral in far-field.

To compare results for the above circular-hole problem, use

$\psi_{in}(\underline{x}) = E_0\, e^{i\underline{k}_0\cdot\underline{x}}$

Result: $\psi(\underline{x}) = \underline{-ik\frac{e^{ikr}}{r}a^2E_0}\;\frac{\cos\alpha + \cos\vartheta}{2\cos\alpha}\;\underline{\frac{J_1(ka\xi)}{ka\xi}}$

(ξ same as above)

Underlined parts identical with vectorial result.

$$\frac{dP}{d\Omega} = \frac{1}{2Z_0}r^2\psi\psi^*, \quad \frac{d\sigma}{d\Omega} = \frac{dP}{d\Omega}\Big/\left[\frac{1}{2Z_0}E_0E_0^*\right]$$

$$\sigma = \int_{\vartheta=0}^{\pi/2}\int_{\varphi=0}^{2\pi}\frac{d\sigma}{d\Omega}\,d\Omega, \quad \text{transmission factor } T = \frac{\sigma}{a^2\pi}$$