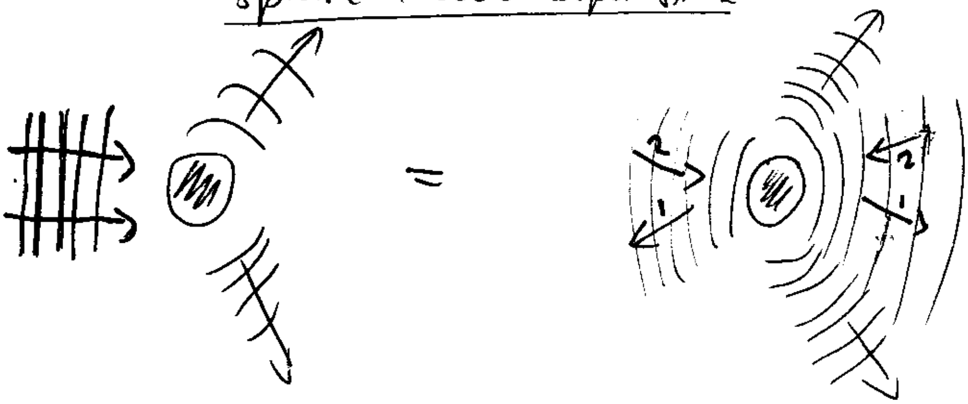
spherical wave expansion



use all spherical waves

$\underline{E}_{in} = \sqrt{2}\,\underline{\varepsilon}_{\pm}\, e^{ikz}$ = incident wave $= \sum \underset{in}{h_l^{(1)}} \cdots + \underset{out}{h_l^{(2)}} \cdots$

scattered wave $= \sum h_l^{(1)}$

SUM $\quad \underset{\substack{in \\ scat}}{\sum\sum} h_l^{(1)} \quad + h_l^{(2)}$

obtain: σ_{sc}, σ_{tot}, σ_{abs}, $\frac{d\sigma_{sc}}{d\Omega}$

Find amplitudes of scattered waves by matching B/C for total field on surface of scatter

good for:
- scatter with ϕ not larger than a few λ
- scatter may be very strong $\left(\frac{d\varepsilon}{\varepsilon} \gtrsim 1\right)$
- Result exact in limit $l \to \infty$

- qm: use at low energy (kc) λ_{db} big)

as opposed to Born approx.
- ϕ quite unlimited
- $\frac{d\varepsilon}{\varepsilon} \ll 1$
- perturbation. Don't need sum, just one $\int$
  Result not exact., may develop higher-order approx.
- qm: use at high energy

Spherical-wave expansion

Expand both incident and scattered waves in $\underline{E}^{(E)}_{(\ell m)}$ and $\underline{E}^{(M)}_{(\ell m)}$ and use props of spherical waves to describe scattering

Incident wave

preliminary: for scalar plane wave $\psi = e^{ikz}$

$$G(\underline{x},\underline{x}') = \frac{e^{ik|\underline{x}-\underline{x}'|}}{4\pi|\underline{x}-\underline{x}'|} = ik\sum_\ell h_\ell^{(1)}(kr')\, j_\ell(kr)\sum_m Y^*_{\ell m}(\vartheta,\varphi)Y_{\ell m}(\vartheta',\varphi') \Big| r'>r$$

flip r, r'

Direction of k is that of $\underline{r}'$

take $r'\to\infty$ on both sides, use $h_\ell^{(1)}(kr')\approx(-i)^{\ell+1}\frac{e^{ikr'}}{kr'}$

$$\frac{e^{ikr'}}{4\pi r'}e^{-ik\hat{\underline{n}}'\cdot\underline{x}} = i k\sum(-i)^\ell j_\ell(kr)\frac{e^{ikr'}}{kr'}\sum Y^*_{\ell m}(\vartheta,\varphi)Y_{\ell m}(\vartheta',\varphi') \Big| cc$$

$$e^{ik\underline{n}'\cdot\underline{x}} = 4\pi\sum_\ell i^\ell j_\ell(kr)\underbrace{\sum_m Y^*_{\ell m}(\vartheta,\varphi)Y_{\ell m}(\vartheta',\varphi')}_{\frac{2\ell+1}{4\pi}P_\ell(\cos\gamma)}$$

$$e^{i\underline{k}\cdot\underline{x}} = \sum_\ell i^\ell(2\ell+1)j_\ell(kr)P_\ell(\cos\gamma) = \sum_\ell i^\ell j_\ell(kr)\sqrt{4\pi(2\ell+1)}\;Y_{\ell 0}(\gamma)$$

wave vector $(k,\vartheta',\varphi') \overset{\text{use}}{\underset{\text{as}}{=}} k\hat{\underline{z}}$

observation point (r,ϑ,φ)

use as $\boxed{e^{ikz} = \sum i^\ell j_\ell(kr)\sqrt{4\pi(2\ell+1)}\;Y_{\ell 0}(\vartheta,\varphi)}$

now generalize to vector field: have polarization choices.

Symmetry: will have spherical scattering center

=> use incident wave with symmetry about z-axis

-> circularly polarized wave $(\underline{\varepsilon}_1 \pm i\underline{\varepsilon}_2) = \ldots = (\hat{\underline{r}}\sin\vartheta + \hat{\underline{\vartheta}}\cos\vartheta \pm i\hat{\underline{\varphi}})e^{\pm i\varphi}$

Due to symmetry, $\underline{\varepsilon}_\pm$-field will only scatter into $\underline{X}_{l\pm1}$ (which also go as $e^{\pm i\varphi}$)

$\Rightarrow$ can simplify math by choosing <u>circ. pol. incident wave</u> (19)

Expand $\left|\begin{array}{l} (\underline{\varepsilon}_1 \pm i\underline{\varepsilon}_2)\, e^{ikz} = \underline{E}(\underline{x}) \\ \underline{B}(\underline{x}) = \frac{1}{i}\hat{\underline{k}}\times\underline{E} \overset{\underline{k}=k\hat{z}}{=} \mp i\,\underline{E}(\underline{x}) \end{array}\right\|$; normalization issue.. (that's field with amplitude $\sqrt{2}$)

$$\boxed{\sum_{lm} \underbrace{a_\pm(l,m)}_{\hat{=}\,a_M(l,m)} j_l(kr)\underline{X}_{lm} + \frac{i}{k}\underbrace{b_\pm(l,m)}_{\hat{=}\,a_E(l,m)}\,\underline{\nabla}\times j_l(kr)\underline{X}_{lm} = \underline{E}(\underline{x})} \quad \Big|\times \underline{X}^*_{l'm'}\,d\Omega$$

b/c we can only have regular functions at $r=0$. orthogonality

$$j_l(kr)\,a_\pm(l,m) = \int \underline{E}(\underline{x})\,\underline{X}^*_{lm}\,d\Omega$$

insert explicit $\underline{E}(\underline{x})$

$$= \int \sum_{l'} i^{l'} j_{l'}(kr)\sqrt{4\pi(2l'+1)}\; Y_{l'0}\;\underbrace{(\underline{\varepsilon}_1 \pm i\underline{\varepsilon}_2)}\; \frac{\hat{\underline{L}}^* Y^*_{lm}}{\sqrt{l(l+1)}}\,d\Omega \qquad l \text{ must be } l'$$

$$a_\pm(l,m) = \frac{i^l\sqrt{4\pi(2l+1)}}{\sqrt{l(l+1)}}\int Y_{l0}\left(L_x^* \pm i L_y^*\right)Y^*_{lm}\,d\Omega$$

$$= \quad '' \quad \int Y_{l0}\left(L_\mp Y_{lm}\right)^* d\Omega = \;''\; \int Y_{l0}\,Y^*_{l\,m\mp1}\,d\Omega \times\sqrt{(l\pm m)(l\mp m+1)}$$

$$= \frac{i^l\sqrt{4\pi(2l+1)}}{\cancel{\sqrt{l(l+1)}}}\;\delta_{0,m\mp1}\,\cancel{\sqrt{(l\pm m)(l\mp m+1)}}$$

$$\boxed{a_\pm(l,m) = i^l\sqrt{4\pi(2l+1)}\,\delta_{m,\pm1}} \quad \hat{=}\; a^M(l,m)$$

- equivalent calculation for $\underline{B}$ yields $b_\pm(l,m) = \mp i\, a_\pm(l,m)$

$$\boxed{\begin{aligned} \underline{E}(\underline{x}) &= \sum_{l=1}^{\infty} i^l\sqrt{4\pi(2l+1)}\left[j_l(kr)\underline{X}_{l,\pm1} \pm \frac{1}{k}\underline{\nabla}\times j_l(kr)\underline{X}_{l,\pm1}\right] \\ c\,\underline{B}(\underline{x}) &= \sum_{1}^{\infty} i^l\sqrt{4\pi(2l+1)}\left[-\frac{i}{k}\underline{\nabla}\times j_l(kr)\underline{X}_{l,\pm1} \mp i\, j_l(kr)\underline{X}_{l,\pm1}\right] \end{aligned}} \;= \sqrt{2}\,\underline{\varepsilon}_\pm e^{ikz}$$

are expansions of p.w. $\underline{E}(\underline{x}) = (\underline{\varepsilon}_1 \pm i\underline{\varepsilon}_2)\,e^{ikz}$; amp $E_0 = \sqrt{2}$

(20)

<u>Symmetry</u> scattered waves will all share azimuthal symmetry of incident wave if scatterer is radially symmetric

(make some counterexamples)

$\Rightarrow$ for $\underline{E}_{in} = (\underline{\varepsilon}_1 \pm i\underline{\varepsilon}_2)e^{ikz}$, scattering will only occur into modes $\propto \underline{X}_{l,\pm1}$ (no mixing)

outgoing

$$\underline{E}_{sc} = \frac{1}{2}\sum i^l \sqrt{4\pi(2l+1)}\Big[\overset{TE}{\alpha_\pm(l)\, h_l^{(1)}(kr)\underline{X}_{l\pm1}} \pm \overset{TM}{\frac{\beta_\pm(l)}{k}\,\underline{\nabla}\times h_l^{(1)}(kr)\underline{X}_{l\pm1}}\Big]$$

F.L. $\underline{B}_{sc} = \frac{1}{i\omega}\underline{\nabla}\times\underline{E}\ \ldots$ (10.57) ; all funny constants for later convenience

[use $\underline{\nabla}\times\underline{\nabla}\times h\underline{X} = \underline{\nabla}(\underline{\nabla}\cdot h\underline{X}) - \Delta h\underline{X} = k^2\underline{X}$]

[$\underline{\nabla}\times h\underline{X}$ just leave]

$\alpha_\pm(l)$ and $\beta_\pm(l)$ tbd. later.

<u>Comparison with 9.122</u>

$$\underline{E}_{sc} = Z_0\sum_l\Big[\frac{i}{k}a_E(l,\pm1)\,\underline{\nabla}\times h_l^{(1)}(kr)\underline{X}_{l\pm1} + a_M(l,\pm1)h_l^{(1)}(kr)\underline{X}_{l\pm1}\Big]$$

$$\boxed{\begin{aligned} a_E(l,\pm1) &= \pm\frac{1}{2Z_0}\, i^{l-1}\sqrt{4\pi(2l+1)}\,\beta_\pm(l) \\ a_M(l,\pm1) &= \frac{1}{2Z_0}\, i^l\sqrt{4\pi(2l+1)}\,\alpha_\pm(l)\end{aligned}}$$ into formalism of Chapter 9

Incident intensity: $I_{inc} = \frac{1}{2Z_0}\underline{E}\cdot\underline{E}^* = \frac{1}{2Z_0}\sqrt{2}^2(\underline{\varepsilon}_\pm\cdot\underline{\varepsilon}_\pm^*) = \frac{1}{Z_0}$

- $\frac{d\sigma}{d\Omega} = \frac{dP}{d\Omega}\cdot\frac{1}{I_{inc}}$ per def. with $\frac{dP}{d\Omega}$ far-field; use $\underline{\nabla}\times \to ik\,\hat{n}\times$

with $$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\Big|\sum(-i)^{l+1}\big[a_E(l,\pm1)\underline{X}_{l,\pm1}\times\hat{n} + a_M(l,\pm1)\underline{X}_{l,\pm1}\big]\Big|^2$$

<u>Eq. 9.150</u> for $m=\pm1$ (and its far field)

insert $a_E, a_M \Rightarrow$

$$\frac{d\sigma}{d\Omega}\Big|_{sc} = \frac{\pi}{2k^2}\Big|\sum_l\Big[\pm\sqrt{2l+1}\,\beta_\pm(l)\,(\hat{n}\times\underline{X}_{l\pm1}) - i\sqrt{2l+1}\,\alpha_\pm(l)\underline{X}_{l,\pm1}\Big]\Big|^2$$

($\times\, i$ under $|\ |$ to get Eq. 10.63.)

$$\sigma_{sc} = \frac{P_{sc}}{I} \quad \text{w/} \quad P_{sc} = \frac{Z_0}{2k^2}\sum_l \left[|a_E(l,\pm 1)|^2 + |a_M(l,\pm 1)|^2\right]$$

and $I = \frac{1}{Z_0}$ Insert a_E, a_M in terms of β, α

$$\Rightarrow \boxed{\sigma_{sc} = \frac{\pi}{2k^2}\sum\left[(2l+1)\left(|\alpha_\pm(l)|^2 + |\beta_\pm(l)|^2\right)\right]} \qquad \text{Eq. 10.61a}$$

Absorption cross section

Use $j_l = \frac{1}{2}(h_l^{(1)} + h_l^{(2)})$ in $\underline{E}_{inc}$ and add to $\underline{E}_{sc}$.

$h_l^{(1)}$: outgoing, $h_l^{(2)}$: ingoing

$\underline{E}_{total} = \underline{E}_{inc} + \underline{E}_{sc} =$

$$= \underbrace{\sum \tfrac{1}{2} i^l \sqrt{4\pi(2l+1)}}_{\text{same prefactors...}} \underline{X}_{l,\pm 1}\Big\{\underset{\text{mag}}{h^{(1)}(kr)\left[1 + a_\pm(l)\right]}\overset{\text{out}}{} + \overset{\text{in}}{h^{(2)}(kr)}\Big\}$$

electric multip.

$$+ \underbrace{\sum \tfrac{1}{2} i^l \sqrt{4\pi(2l+1)}}_{\text{... same prefactor as in expansion of } E_{sc} \text{ and } E_{in}} \left(\pm\frac{1}{k}\right)\underline{\nabla}\times\Big\{\underset{\text{out}}{h_l^{(1)}\left[1+\beta_\pm(l)\right]} + \underset{\text{in}}{h_l^{(2)}}\Big\}\underline{X}_{l,\pm 1}$$

(to be used later)

and $\underline{B}_{tot} = \frac{1}{i\omega}\underline{\nabla}\times\underline{E}$ w/ $\underline{\nabla}\times\underline{\nabla}\times\underline{X} = k^2\underline{X}$

Total absorption = all in − all out =

$$\frac{\pi}{2k^2}\sum_l (2l+1)\Big[\underset{\text{in}}{2} - \underset{\text{out } M}{|1+a_\pm(l)|^2} - \underset{\text{out } E}{|1+\beta_\pm(l)|^2}\Big] = \sigma_{abs} \qquad 10.61b$$

Total abs + scattering

$$\sigma_t = \sigma_{abs} + \sigma_{sc} = + \frac{\pi}{2k^2}\sum_l (2l+1)\left[2 - (1+a)(1+a^*) - (1+b)(1+b^*) + aa^* + bb^*\right]$$

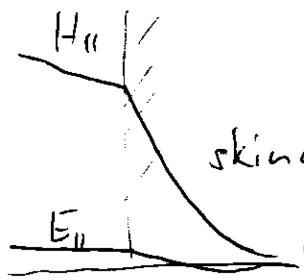
$$\boxed{\sigma_t = -\frac{\pi}{k^2}\sum_l (2l+1)\left[\mathrm{Re}(\alpha_\pm(l)) + \mathrm{Re}(\beta_\pm(l))\right] \qquad 10.62}$$

__Task__: Find $\alpha_\pm(l)$, $\beta_\pm(l)$ by matching B/C for total fields on scatterer.

Hard but o.k. if scatter spherical and $\underline{E}_{tan} = Z_s \, \underline{n} \times \underline{B} \frac{1}{\mu_0}$

surface impedance Z_s
$[Z_s] = \Omega$

- conductor: $Z_s = 0$
- conductor with loss



$\delta = \sqrt{\frac{2}{\sigma_c \mu_c \omega}}$

see Eqn. 8.11

$$\Rightarrow Z_s = (1-i)\sqrt{\frac{\omega \mu_c}{2\sigma_c}} = \frac{1-i}{\sigma \delta}$$

- superconductor: $Z_s = -i\omega\mu_0\lambda_L$

London penetration depth of B-field (has significance similar to δ)

__way to find α, β:__ $\underline{E}_{tan} = \hat{\underline{n}} \times (\underline{E}_{total} \times \hat{\underline{n}}) = \frac{Z_s}{\mu_0} (\hat{\underline{n}} \times \underline{B}_{total})$

types of terms:
- $\hat{\underline{n}} \times \underline{X}_{lm}$
- $\hat{\underline{n}} \times (\underline{X}_{lm} \times \hat{\underline{n}}) = \underline{X}_{lm}$ (transverse $\underline{X}$ b/c)

Eq. 10.60
- $\hat{\underline{n}} \times (\underline{\nabla} \times f \underline{X}) = \left[\frac{1}{r}\frac{d}{dr} r f(r)\right] \hat{\underline{n}} \times (\hat{\underline{n}} \times \underline{X}) = -\frac{1}{r}\left[\frac{d}{dr} r f(r)\right] \underline{X}_{lm}$
- $\hat{\underline{n}} \times \left[\hat{\underline{n}} \times (\nabla \times f\underline{X})\right] = -\left[\frac{1}{r}\frac{d}{dr} r f(r)\right] \hat{\underline{n}} \times \underline{X}_{lm}$

In result, only terms $\sim \underline{X}_{lm}$ and $\sim \hat{\underline{n}} \times \underline{X}_{lm}$

Equate coefficients.

$\boxed{\underline{E}_{tan} = \hat{\underline{n}} \times (\underline{E} \times \hat{\underline{n}})}$ use $\hat{\underline{n}} \times (\underline{X}_{lm}) \times \hat{\underline{n}} = \underline{X}_{lm}$ (b/c X_{lm} transverse)

$$\hat{\underline{n}} \times \left[(\underline{\nabla} \times f \underline{X}) \times \hat{\underline{n}}\right] = \hat{\underline{n}} \times \left[\frac{1}{r}\frac{d}{dr} f \underline{X}\right] \qquad \text{(from Eqn. 10.60)}$$

Thus, $\underline{E}_{tan} = \sum \frac{1}{2} i^{l} \sqrt{4\pi(2l+1)} \left\{h^{(1)}(ka)[1+a_{\pm}(l)] + h^{(2)}(ka)\right\} \underline{X}_{l\pm 1}$

$$+ \sum \frac{1}{2} i^{l} \sqrt{4\pi(2l+1)} \left(\pm \frac{1}{k}\right) \hat{\underline{n}} \times \left[\frac{1}{r}\partial_r \left\{h_l^{(1)}(kr)[1+\beta_{\pm}(l)] + h_l^{(2)}(kr)\right\}\right]_{r=a} \underline{X}_{l}$$

note $x = kr$ in 10.64

Total fields

$$c\underline{B} = \sum_l \frac{i^l}{2} \sqrt{4\pi(2l+1)} \frac{1}{ik} \underline{\nabla} \times \left\{h_l^{(1)}(kr)[1+a_{\pm}(l)] + h_l^{(2)}(kr)\right\} \underline{X}_{l,\pm 1}$$

$$\mp \sum_l \frac{i^l}{2} \sqrt{4\pi(2l+1)}\; i \left\{h_l^{(1)}(kr)[1+\beta_{\pm}(l)] + h_l^{(2)}(kr)\right\} \underline{X}_{l,\pm 1}$$

$$c\,\underline{n} \times \underline{B} = \sum_l \frac{i^l}{2} \sqrt{4\pi(2l+1)} \frac{i}{kr} \left[\frac{\partial}{\partial r} r \left\{h_l^{(1)}(kr)[1+a_{\pm}(l)] + h_l^{(2)}(kr)\right\}\right]_{r=a} \underline{X}_{l,\pm 1}$$

$$\mp \sum_l \frac{i^l}{2} \sqrt{4\pi(2l+1)}\; i \left\{h_l^{(1)}(ka)[1+\beta_{\pm}(l)] + h_l^{(2)}(kr)\right\} \hat{\underline{n}} \times \underline{X}_{l,\pm 1}$$

$Z_s\, \hat{\underline{n}} \times \frac{\underline{B}}{\mu_0} = \underline{E}_{tan}$: Equate coefficient of $X_{l,\pm 1}$: use $\frac{Z_s}{\mu_0 c} = \frac{Z_s}{Z_0}$

B/c for total fields

$$\boxed{1 + a_{\pm}(l) = \frac{i \frac{Z_s}{Z_0} \frac{1}{kr}\left[\frac{d}{dr} r h^{(2)}(kr)\right]_{r=a} - h^{(2)}(ka)}{h^{(1)}(ka) - i\frac{Z_s}{Z_0}\frac{1}{kr}\left[\frac{d}{dr} r h^{(1)}(kr)\right]_{r=a}}}$$

$\beta_{\pm}(l) = $ (replace $\frac{Z_s}{Z_0}$ by $\frac{Z_0}{Z_s}$)

note $x = kr$ and

$$\frac{1}{kr}\frac{d}{dr} r h(kr) = \frac{1}{x}\frac{d}{dx} x h(x)$$

case of interest: $ka \ll 1 \Rightarrow h_\ell^{(1)} = \frac{1}{2}(j_\ell^{(1)} + i n_\ell)$

$$\simeq +\frac{1}{2} i n_\ell = -\frac{i}{2}\frac{(2\ell-1)!!}{(ka)^{\ell+1}}$$

$$h_\ell^{(2)} \simeq \frac{i}{2}\frac{(2\ell-1)!!}{(ka)^{\ell+1}}$$

- get Eqn. 10.69, $a_\pm \propto (ka)^{2\ell+1}$
- only significant terms are $a_\pm$ $(\ell = 1)$

$Z_s = 0$: see p. 477 in book (conducting sphere)

get Eqn. 10.16 with explicitly calculated ε and $\frac{m}{c}$

Other cases No damping ($Z_s = 0, \infty$, pure imaginary)

$$|1+\alpha| = |1+\beta| = 1$$

$$\alpha_\pm = e^{2i\delta_\ell} - 1 \qquad \beta_\pm = e^{2i\delta_\ell'} - 1$$

scattering phase shifts

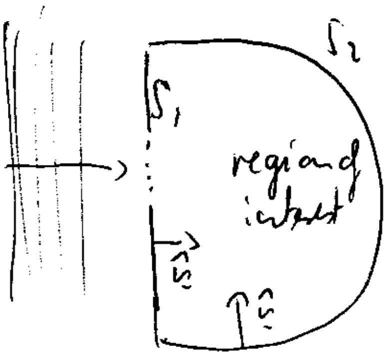
Note: in expressions for total fields*, the amplitudes of the outgoing wave go as $(\alpha_\pm + 1)$ and $(\beta_\pm + 1)$ hence name "phase shifts"

(explicit expressions for δ_ℓ, δ_ℓ' see book)

* $\underline{E}_{total} = \sum \frac{1}{2} i^\ell \sqrt{4\pi(2\ell+1)}\, \underline{X}_{\ell,\pm 1} \left\{ h_\ell^{(1)}(kr) e^{2i\delta_\ell} + h_\ell^{(2)}(kr) \right\} +$

$$\pm \sum \frac{1}{2} i^\ell \sqrt{4\pi(2\ell+1)}\, \frac{1}{k} \underline{\nabla} \times \left\{ h_\ell^{(1)}(kr) e^{2i\delta_\ell'} + h_\ell^{(2)}(kr) \right\}$$

Diffraction theory



scalar theory

$$(\underline{\nabla}^2 + k^2)\psi(\underline{x}) = 0$$

$$(\underline{\nabla}^2 + k^2)\, G(\underline{x},\underline{x}') = -\delta(\underline{x}-\underline{x}')$$

1st Green: $$\int_V (\psi\Delta\varphi - \varphi\Delta\psi)\, d^3x = \int_{\partial V} \left(\psi\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial\psi}{\partial n}\right) da$$

$\hat{n}$ outward

Use $\psi = \psi$ and $\varphi = G(\underline{x},\underline{x}')$, and use $\hat{n}$ inward

$$\int \left(\psi(\underline{x}')\,\Delta_{x'} G(\underline{x},\underline{x}') - G(\underline{x},\underline{x}')\,\Delta_{x'}\psi(\underline{x}')\right) d^3x' =$$

$$= \int_V \left\{\psi(\underline{x}')\left(-\delta(\underline{x}-\underline{x}') - \cancel{k^2 G(\underline{x},\underline{x}')}\right) + \cancel{G(\underline{x},\underline{x}')k^2}\right\} d^3x' = -\psi(\underline{x}) =$$

$$\boxed{= \int_{\partial V} \left\{G(\underline{x},\underline{x}')\frac{\partial\psi(\underline{x}')}{\partial n'} - \psi(\underline{x}')\frac{\partial}{\partial n'}G(\underline{x},\underline{x}')\right\} da' = -\psi(\underline{x})}$$

$\hat{n}'$ inward

Kirchhoff's approach

Use free-space Green's function $G(\underline{x},\underline{x}') = \frac{1}{4\pi}\frac{e^{ikR}}{R}$; $R = |\underline{x}-\underline{x}'|$

$$\psi(\underline{x}) = -\frac{1}{4\pi}\int_{\partial V} \left\{\frac{e^{ikR}}{R}\,\underline{\nabla}'\psi(\underline{x}') - \psi(\underline{x}')\,\underline{\nabla}' G(\underline{x},\underline{x}')\right\}\hat{n}'\, da'$$

$$\underline{\nabla}' G(\underline{x},\underline{x}') = -\frac{e^{ikR}}{R}\, ik\left(1+\frac{i}{kR}\right)\frac{\underline{R}}{R}$$

$$\boxed{\psi(\underline{x}) = -\frac{1}{4\pi}\int_{S_1+S_2} \frac{e^{ikR}}{R}\left\{\underline{\nabla}'\psi(\underline{x}') + ik\left(1+\frac{i}{kR}\right)\frac{\underline{R}}{R}\,\psi(\underline{x}')\right\}\hat{n}'\, da'}$$

Further: let $S_2 \to \infty$. Then, ($G \approx \frac{e^{ikr'}}{4\pi r'}$ and) $\psi(\underline{x}') \approx f(\vartheta', \varphi') \frac{e^{ikr'}}{r'}$ (26)

(and semicircular)

Insert this $\psi(\underline{x}')$ into above, find $\int_{S_2} \cdot da' \sim \int_{S_2} \frac{e^{ikr'}}{r'} f(\vartheta',\varphi') \frac{e^{ikr'}}{r'^2} \underbrace{\frac{R}{R}\cdot\hat{n}'}_{=1} da$

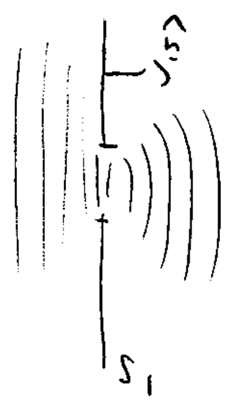
$$\sim \int_{S_2} \frac{e^{2ikr'}}{r'^3} f(\vartheta',\varphi')\, r'^2\, d\Omega' \sim \frac{1}{r'} \to 0 \text{ for } r' \to \infty$$

So: $$\boxed{\psi(\underline{x}) = -\frac{1}{4\pi}\int_{S_1} \frac{e^{ikR}}{R}\left\{\underline{\nabla}'\psi(\underline{x}') + ik\left(1+\frac{i}{kR}\right)\frac{\underline{R}}{R}\,\psi(\underline{x}')\right\}\cdot\hat{\underline{n}}'\,da'}$$

$\hat{n}'$ inward, $\underline{R} = \underline{x} - \underline{x}'$ (wrt volume of interest)

Kirchhoff's diffraction integral

Physical intuition (Kirchhoff approximation)



- $\psi, \frac{\partial\psi}{\partial n} = 0$ on S_1, except in holes
- $\psi, \frac{\partial\psi}{\partial n}$ in holes of S_1 are replaced by the values under entire absence of S_1

side issue

Inconsistency: 1st condition $\psi, \frac{\partial\psi}{\partial n} = 0$ on any surface

$\Rightarrow \psi(\underline{x}) \equiv 0$ everywhere.

[See Laplace:  $\Phi = 0$ and $\underline{E} = 0$ $\Rightarrow \Phi = 0$ everywhere]

fix: choose G_D or G_N : $(\nabla^2 + k^2)\, G_{D,N} = -\delta(\underline{x}-\underline{x}')$

and $G_D = 0$ on S_1 without any holes

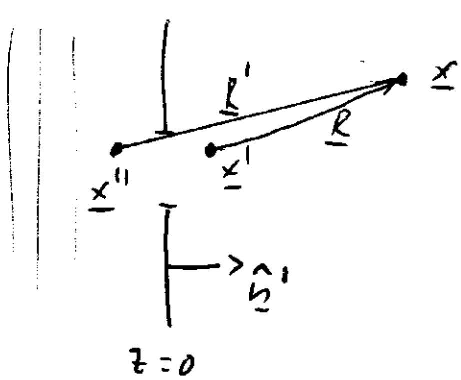
or $\frac{\partial}{\partial n} G_N = 0$ on S_1 " " "

then: Dirichlet: $\psi_D(\underline{x}) = \int \psi(\underline{x}') \frac{\partial G_D(\underline{x},\underline{x}')}{\partial n'}\, da'$

$$\psi_N(\underline{x}) = -\int G_N(\underline{x},\underline{x}') \frac{\partial}{\partial n'} \psi(x')\, da'$$

We then only need to require that either $\psi(\underline{x}')$ or $\frac{\partial}{\partial n'}\psi(\underline{x}')$ on S_1 vanishes. Inconsistency gone.

Example screen with holes at $z=0$



$$G_K = \frac{e^{ikR}}{4\pi R}$$

$$G_{\binom{D}{N}} = \frac{1}{4\pi}\left(\frac{e^{ikR}}{R} \mp \frac{e^{ikR'}}{R'}\right)$$

with $\underline{R} = \underline{x} - \underline{x}'$
$\underline{R}' = \underline{x} - \underline{x}''$

$\left[\ \psi_D(\underline{x}) = \int \psi(\underline{x}') \frac{\partial G_D}{\partial n'}(\underline{x},\underline{x}')\, da', \quad \psi_N = -\int G_N \frac{\partial}{\partial n'}\psi\, da'\ \right]$

find

$$\boxed{\begin{aligned} \psi_D(\underline{x}) &= \frac{k}{2\pi i}\int_{S_1} \frac{e^{ikR}}{R}\left(1+\frac{i}{kR}\right)\psi(\underline{x}')\, \frac{\underline{R}}{R}\cdot\hat{\underline{n}}'\, da' \\ \psi_N(\underline{x}) &= -\frac{1}{2\pi}\int_{S_1} \frac{e^{ikR}}{R}\, \underline{\nabla}'\psi(\underline{x}')\cdot\hat{\underline{n}}'\, da' \end{aligned}}$$

doubles 2nd term in ψ_K
removes 1st " " "

doubles 1st " "
removes 2nd " "

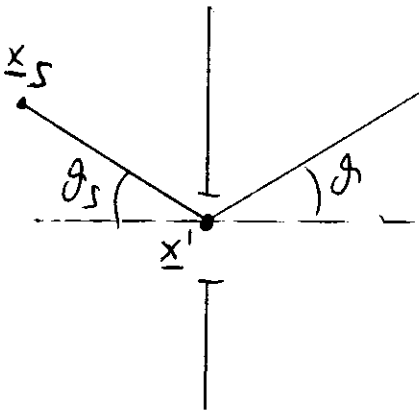
Numerically, ψ_K, ψ_D, ψ_N quite similar. No practical gain (only ideological gain)
Practice: use $\psi_K(\underline{x})$

Sample calculation that exhibits similarity

- $\frac{1}{r}$ amplitude decrease
- phases kr

contains most of the physics

No plane wave, but point source on left side



$$\psi(\underline{x}) = \frac{k}{2\pi i} \int_{S_1} \overbrace{\frac{e^{ik|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|} \; \frac{e^{ik|\underline{x}_S-\underline{x}'|}}{|\underline{x}_S-\underline{x}'|}}^{\text{contains most of the physics}} \mathcal{O}(\vartheta,\vartheta_S)\, da'$$

(apertures only)

obliquity factor: $\mathcal{O}(\vartheta,\vartheta') =$

		specify
$\frac{1}{2}(\cos\vartheta+\cos\vartheta_S)$	for K	ψ, and $\frac{\partial\psi}{\partial n}$
$\cos\vartheta_S$	for N	$\partial\psi/\partial n$
$\cos\vartheta$	for D	ψ
		on S_1

all ~ 1 in practice