| on waveguides, cavities
| on radiation (of multipoles, ...)
| on scattering (No diffraction.)

$\boxed{\text{Types of problems}}$

Waveguides: Types of modes

TEM: $\underline{E}_t = -\nabla_t \Phi_t$ with $\nabla_t^2 \Phi_t = 0$

$\underline{B} = \pm \frac{1}{c} \hat{z} \times \underline{E} = \pm \sqrt{\mu\varepsilon}\, \hat{z} \times \underline{E}$ (for $e^{\pm ikz}$)

$k = \frac{\omega}{c} = \sqrt{\mu\varepsilon}\, c$ (as normal plane wave)

(exist only with center conductor)

TE, TM: solve 2D Helmholtz eqn.

$(\nabla_t^2 + \gamma^2)\psi = 0$ + B/C ; $\gamma^2 = \mu\varepsilon\omega^2 - k^2$

TM: $\psi = E_z$; $\psi = 0$ on S

TE: $\psi = H_z$; $\frac{\partial \psi}{\partial n} = 0$ on S

$$\boxed{\begin{array}{l} \underline{E} \perp \text{walls} \\ \underline{H} \parallel \text{walls} \end{array}}$$

- solutions $\{\psi_i\}$, $\{\gamma_i^2\}$ $(\gamma_i^2 > 0)$
- universal shape of dispersion relation, $k_i = \sqrt{\mu\varepsilon}\sqrt{\omega^2 - \omega_\lambda^2}$

  $v_p v_g = c^2 = \frac{1}{\varepsilon\mu}$ $\quad \omega_i = \frac{\gamma_i}{\sqrt{\mu\varepsilon}}$ cutoff frequency

other fields: TM: $\underline{E}_t = \pm \frac{ik}{\gamma^2} \nabla_t E_z$, $\underline{H}_t = \frac{\pm 1}{Z} \hat{z} \times \underline{E}_t$, $Z = \frac{k}{\varepsilon\omega}$

TE: $\underline{H}_t = \pm \frac{ik}{\gamma^2} \nabla_t H_z$, $\underline{E}_t = \mp Z\, \hat{z} \times \underline{H}_t$; $Z = \frac{\mu\omega}{k}$

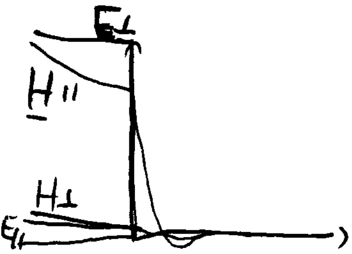
Typical methods: separation of variables (cartesian, cylindrical ..)

Power: $P = \int \mathrm{Re}\left[\hat{z} \cdot \frac{1}{2}(\underline{E} \times \underline{H}^*)\right] da$

$$= \frac{1}{2\sqrt{\mu\varepsilon}} \left(\frac{\omega}{\omega_i}\right)^2 \sqrt{1 - \frac{\omega_i^2}{\omega^2}} \begin{Bmatrix} \varepsilon \\ \mu \end{Bmatrix}_{TE}^{TM} \int \psi^* \psi \, da \qquad \psi = \begin{Bmatrix} E_z \\ H_z \end{Bmatrix}$$

energy density: $U = \frac{P}{v_g}$ with $v_g = \frac{d\omega_i}{dk}$; $U = \frac{1}{2}\left(\frac{\omega}{\omega_i}\right)^2 \begin{Bmatrix} \varepsilon \\ \mu \end{Bmatrix}_{TE}^{TM} \int \psi \psi^* da$

Waveguide losses



$$\delta = \sqrt{\frac{2}{\mu_c \omega \sigma}}$$

$$\underline{E}_\parallel = \sqrt{\frac{\mu_c \omega}{2\sigma}}\,(1-i)(\underline{n} \times \underline{H}_\parallel) \quad \text{on surface}$$

$$\boxed{\frac{dP}{da} = \left|\frac{1}{2}\,\mathrm{Re}\left[\underline{n} \cdot \underline{E} \times \underline{H}^*\right]\right| = \frac{\mu_c \omega \delta}{4}\,|\underline{H}_\parallel|^2 = \frac{1}{2\sigma\delta}|\underline{H}_\parallel|^2 \quad \text{into wall}}$$

in wg: $\displaystyle \frac{dP}{dz} = -\frac{1}{2\sigma\delta}\oint |\underline{H}_\parallel|^2\, dl$

$$P = P_0\, e^{-2\beta z} \quad \text{with absorption coefficient } \beta = -\frac{1}{2P}\frac{dP}{dz}$$

Explicit calculation: Get $\psi \Rightarrow P \propto \int \psi^* \psi \, da$
Get $\underline{H}_\parallel \Rightarrow \frac{dP}{dz} \propto \oint |\underline{H}_\parallel|^2 dl$ $\Rightarrow \beta$

$k = k_0 + i\beta + \alpha$; find in non degenerate cases $\alpha = \beta$ $(\ll k_0)$

by perturbation of boundary conditions.

Review problems 8.12, 8.13 (degenerate perturbation theory)

typical: $(\nabla_t^2 + \gamma_0^2)\psi_0 = 0 \;\wedge\; \psi_0 = 0$ on S (TM)

perturbed: $(\nabla_t^2 + \gamma^2)\psi = 0 \;\wedge\; \psi = f \frac{\partial \psi_0}{\partial n}$ on S

Green II $\Rightarrow$ $\displaystyle (\gamma^2 - \gamma_0^2) = -f\, \frac{\oint \left|\frac{\partial \psi_0}{\partial n}\right|^2 dl}{\int |\psi_0|^2 da}$ $(\sim$ if $f = f(x,y))$

$$f = (1+i)\,\frac{\mu_c \delta}{2\mu}\left(\frac{\omega}{\omega_0}\right)^2 \;;\quad \omega_0 = \frac{\gamma_0}{\sqrt{\mu\epsilon}}$$

for damping of TM modes.

<u>Cavities</u>: Form <u>standing waves</u> $\sim e^{\pm ikz}$ to match B/C on endpieces.

TM: $E_t = 0$ on ends ;

TE: $H_z = 0$ on ends ,

$$\omega = \frac{1}{\sqrt{\mu\varepsilon}} \sqrt{\gamma_i^2 + \left(\frac{p\pi}{d}\right)^2} \; ; \qquad p = \begin{cases} 0, 1, 2 \ldots & TM \\ 1, 2, \ldots & TE \end{cases}$$

(equs. see page 369)

<u>Q-factor</u>:

$$Q = \omega_0 \frac{U}{dU/dt} \; ; \qquad \text{FWHM of } P(\omega) = \frac{\omega_0}{Q} \; , \quad P(\omega) \sim |E(\omega)|^2$$

$$U = \int u \, d^3x \; ; \quad u = \frac{1}{4}\varepsilon \underline{E}\cdot\underline{E}^* + \frac{1}{4}\mu \underline{H}\cdot\underline{H}^* ;$$

$$Q \Leftarrow \quad = \frac{d}{4} \begin{pmatrix} \varepsilon \\ \mu \end{pmatrix}_{TE}^{TM} \left[1 + \left(\frac{p\pi}{\gamma_i d}\right)^2\right] \int_A |\psi|^2 \, da \qquad (\times 2 \text{ for } p = 0)$$

$$P = \frac{dU}{dt} = \frac{1}{2\delta\sigma} \int_{\text{all surfaces.}} |\underline{H}_{\parallel}|^2 \, da$$

<u>Frequency shift</u>: $\Delta\omega = -\frac{\omega_0}{2Q} = -\frac{1}{2} \times \text{FWHM}$

Normal modes.

For both TE, TM we define

$$\underline{E}_i^+ = [\underline{E}_i(x,y) + E_{z,i}(x,y)\,\hat{\underline{z}}]\, e^{ik_i z}$$

$$\underline{H}_i^+ = \text{(analogous)}$$

$$\underline{E}_i^- = [\underline{E}_i(x,y) - E_{z,i}(x,y)\,\hat{\underline{z}}]\, e^{-ik_i z}$$

$$\underline{H}_i^- = [-\underline{H}_i(x,y) + H_{z,i}(x,y)\,\hat{\underline{z}}]\, e^{-ik_i z}$$

where E_z and H_z are the formerly called "ψ - fields" and $\underline{E}_i$ and $\underline{H}_i$ the corresponding transverse fields. The total mode fields are the right-propagating $\underline{E}_i^+$, $\underline{H}_i^+$ and left-propagating $\underline{E}_i^-$, $\underline{H}_i^-$. Normalization via

$$\int \underbrace{\underline{E}_i \cdot \underline{E}_j}_{\text{transverse fields}}\, da = \delta_{ij} \qquad \text{Note Eqns. 8.131 - 8.134}$$

Main application:

Outside a localized source with $\underline{J}(\underline{x})$ inside a waveguide, the field propagating to the right (left) is

$$\underline{E} = \sum_i A_i^\pm \underline{E}_i^\pm$$

with the above defined mode fields and amplitudes $A_i^\pm$. It is

$$A_i^\pm = -\frac{Z_i}{2}\underbrace{\int}_{\substack{\text{source}\\ \text{volume}}} \underline{J}(\underline{x}) \cdot \underbrace{\underline{E}_i^\mp(\underline{x})}_{\substack{\text{total mode field}\\ \text{including } e^{\mp ik_i z}}} d^3x \qquad Z_i = \begin{cases} \frac{k}{\varepsilon\omega} & ; \text{TM} \\ \frac{\mu\omega}{k} & ; \text{TE} \end{cases}$$

Radiation:

$$\underline{A}(\underline{x}) = \frac{\mu_0}{4\pi}\int \underline{J}(\underline{x}')\frac{1}{|\underline{x}-\underline{x}'|}e^{ik|\underline{x}-\underline{x}'|}\,d^3x'$$

$(\rho, \underline{J}) \sim e^{-i\omega t}$

exact calculation sometimes possible

then $\underline{H} = \frac{1}{\mu_0}\underline{\nabla}\times\underline{A}$

$\underline{E} = \frac{i}{k}Z_0\,\underline{\nabla}\times\underline{H}$ from M. A.

Cartesian expansions

small source, $d \lesssim \lambda$, $\frac{1}{r}e^{ikr}\underbrace{e^{-ik\underline{n}\cdot\underline{x}'}}_{\simeq 1}$

$\longrightarrow \underline{p} = \int \underline{x}\,\rho(\underline{x})\,d^3x$

$$\underline{A}_{E1} = -i\omega\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\underline{p}$$

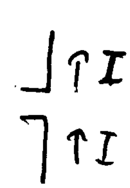
($\Rightarrow \underline{E}, \underline{H}$ for all zones)

Profile: $\frac{dP}{d\Omega}$ in far zone only, $\frac{dP}{d\Omega} = r^2\,\mathrm{Re}\left[\frac{1}{2}(\underline{E}\times\underline{H}^*)\cdot\hat{\underline{n}}\right]$

use $\underline{\nabla}\times = ik\hat{\underline{n}}\times$

$$\frac{dP}{d\Omega} = \frac{c^2k^4Z_0}{32\pi^2}\left|\underbrace{(\hat{\underline{n}}\times\hat{\underline{p}})\times\hat{\underline{n}}}_{\text{direction of } \underline{E}}\right|^2$$

Examples: linear antenna (short). $\rho = \pm\frac{2iI_0}{\omega d}$



R_{rad} via $P = \frac{1}{2}R|I_0|^2$

<u>M1, E2</u> : expand $e^{ik\hat{n}\cdot\underline{x}'} = 1 - ik\hat{n}\cdot\underline{x}'$

$$\frac{e^{ikr}}{|\underline{x}-\underline{x}'|} = \frac{e^{ikr}}{r}\Big(\underbrace{1-\frac{1}{ikr}}_{\text{usually not important}}\Big)$$

$$\underline{A}_* \propto \int \underline{J}(\underline{x}')(-ik\hat{n}\cdot\underline{x}')\,d^3x'$$

= symm. part + antisymm. part

symm. part $\downarrow$ E2

important:

$\underline{\underline{Q}}\cdot\hat{\underline{n}}$ takes role of $\underline{p}$

$$Q_{\alpha\beta} = \int (3x_\alpha x_\beta - r^2\delta_{\alpha\beta})\,\rho(\underline{x})\,d^3x$$

antisymm. part $\downarrow$ M1

$$\underline{m} = \frac{1}{2}\int (\underline{x}\times\underline{J})\,d^3x$$

fields via $\underline{p} \to \frac{\underline{m}}{c}$

$$\underline{E}_{M1} = -Z_0\,\underline{H}_{E1}$$

$$\underline{H}_{M1} = \frac{\underline{E}_{E1}}{Z_0}$$

Spherical multipole moments

$(\nabla^2+k^2)\,\psi(\underline{x})$ 3D, $k=\frac{\omega}{c}$, solved by

$$\psi = \sum_{l,m}\Big[A^{(1)}_{lm}h^{(1)}_l(kr) + A^{(2)}_{lm}h^{(2)}_l(kr)\Big]Y_{lm}(\vartheta,\varphi)$$

out. in

$$h^{(1)}_l \simeq (-i)^{l+1}\frac{e^{ix}}{x}$$

$$h^{(1)}_l = j_l + i n_l$$

$$j_l = J_{l+\frac{1}{2}}(x)\sqrt{\frac{\pi}{2x}}\quad,\ \text{same for } n_l$$

$$G(\underline{x},\underline{x}') = \frac{e^{ik|\underline{x}-\underline{x}'|}}{4\pi|\underline{x}-\underline{x}'|} = ik\sum_{l=0}^{\infty} j_l(kr_<)\,h^{(1)}_l(kr_>)\sum_{m=-l}^{l}Y_{lm}(\vartheta,\varphi)\,Y^*_{lm}(\vartheta',\varphi')$$

where $(\nabla^2 + k^2) G = -\delta(\underline{x}, \underline{x}')$

$$(\nabla^2 + k^2)\psi = -f(\underline{x})$$

$$\psi(x) = \int f(\underline{x}') G(\underline{x}, \underline{x}') d^3x'$$

..... use to identify <u>spherical normal mode</u>

<u>magnetic = TE</u> ; use $\psi = \underline{r} \cdot \underline{H}^M_{lm}$

$$\underline{E}^M_{lm} = Z_0 f^M_{lm}(kr) \hat{\underline{L}} Y_{lm} ,$$

$$\hat{\underline{L}} = \frac{1}{i} \underline{r} \times \underline{\nabla}$$

$$\underline{X}_{lm} := \frac{1}{\sqrt{l(l+1)}} \hat{\underline{L}} Y_{lm}$$

$$\underline{H}^M_{lm} = -\frac{i}{kZ_0} \underline{\nabla} \times \underline{E}^M_{lm} \quad \text{(from F.L.)}$$

electric = TM ; use $\psi = \underline{r} \cdot \underline{E}^{(E)}_{lm}$

$$\underline{H}^E_{lm} = f^E_{lm}(kr) \hat{\underline{L}} Y_{lm}$$

$$\underline{E}^E_{lm} = \frac{iZ_0}{k} \underline{\nabla} \times \underline{H}^E_{lm} \quad \text{(from M.A.L)}$$

- There, f_{lm} are in general superpositions of any spherical Bessel functions, for instance $f_{lm} = A^{(1)}(l,m) h^{(1)}_l(kr) + A^{(2)}(l,m) h^{(2)}_l(kr)$.
- In case it is known that there is only outward radiation, it is $f_{lm}(kr) = A(l,m) h^{(1)}_l(kr)$.

For a given source, the (outgoing) fields are expanded as

$$\underline{H} = \sum_{\ell m} \left[a_E(\ell,m)\, h_\ell^{(1)}(kr)\, \underline{X}_{\ell m} - \frac{i}{k}\, a_M(\ell,m)\, \underline{\nabla} \times h_\ell^{(1)}(kr)\, \underline{X}_{\ell m} \right]$$

$$\underline{E} = \sum_{\ell m} Z_0 \left[\frac{i}{k}\, a_E(\ell,m)\, h_\ell^{(1)}(kr)\, \underline{\nabla} \times h_\ell^{(1)}(kr)\, \underline{X}_{\ell m} + a_M(\ell,m)\, h_\ell^{(1)}(kr)\, \underline{X}_{\ell m} \right]$$

<u>Radiation zone</u>: set $h^{(1)}(kr) = (-i)^{\ell+1} \frac{e^{ikr}}{kr}$ and $\underline{\nabla}\times = ik\,\hat{\underline{n}}\times$

To find the coefficients a_E, a_M, use

Eqns. 9.167 f for sources with diameter $d \gtrsim \lambda$

Eqns. 9.169 - 9.172 for small sources $(d \ll \lambda)$

CH 10 Summary

$$\frac{d\sigma}{d\Omega} = \frac{r^2 |\underline{\varepsilon}^* \cdot \underline{E}_{sc}|^2}{|\varepsilon_0 E_{in}|^2} = \frac{d\sigma}{d\Omega}(\underline{\varepsilon}_0, \underline{\varepsilon}^*)$$

$$\frac{d\sigma}{d\Omega} = \sum_{\underline{\varepsilon}^*} \left\langle \frac{d\sigma}{d\Omega}(\underline{\varepsilon}_0, \underline{\varepsilon}^*) \right\rangle_{\underline{\varepsilon}_0}$$; sum and average over polarizations

dipole scattering $\underline{E}_{sc}$ solely due to induced el. and mag. dipoles

$$\underline{p} = 4\pi\varepsilon_0 \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right) a^3 \underline{E}_{inc}$$

$$\underline{m} = -2\pi a^3 \underline{H}_{in}$$; Rayleigh scattering, $\sim k^4$

Perturbation theory

$$\underline{D} = \underline{D}^{(0)} + \frac{e^{ikr}}{r}\underline{A}_{sc}$$; D_0 formal solution of 3D Helmholtz Eq using $G = e^{ik|\underline{x}-\underline{x}'|}/4\pi|\underline{x}-\underline{x}'|$

$$\frac{\underline{\varepsilon}^* \cdot \underline{A}^{(1)}_{sc}}{D_0} = \frac{k^2}{4\pi}\int d^3x \, e^{iqx}\left(\underline{\varepsilon}^* \cdot \underline{\varepsilon}_0 \frac{\delta\varepsilon(x)}{\varepsilon_0} + (\hat{\underline{n}} \times \underline{\varepsilon}^*)(\hat{\underline{n}}_0 \times \underline{\varepsilon}_0)\frac{\delta\mu_0(\underline{x})}{\mu_0}\right) d^3x$$

$$\underline{q} = \underline{k}_0 - \underline{k} = k(\hat{\underline{n}}_0 - \hat{\underline{n}})$$

$$\frac{d\sigma}{d\Omega} = \frac{|\underline{\varepsilon}^* \cdot \underline{A}^{(1)}_{sc}|^2}{|D_0|^2}$$; good when λ in scatterer $\approx \lambda$ in surrounding medium (as in QM, where $E_{kin} \gg |V|$)

Spherical-wave expansion

for spherical symmetry. Use σ^+ and σ^- - polarized incident waves,

$$\underline{E}_{in} = (\underline{\varepsilon}_1 \pm i\underline{\varepsilon}_2)e^{ikz} \sum_l^{\infty} i^l \sqrt{4\pi(2l+1)}\left[j_l(kr)\underline{X}_{l,\pm1} \pm \frac{1}{k}\underline{\nabla} \times j_l(kr)\underline{X}_{l,\pm1}\right]$$

$$\underline{E}_{sc} = \frac{1}{2}\sum_l^{\infty} i^l \sqrt{4\pi(2l+1)}\left[\alpha_\pm(l) h_l^{(1)}(kr)\underline{X}_{l,\pm1} \pm \frac{\beta_\pm(l)}{k}\underline{\nabla} \times h_l^{(1)}(kr)\underline{X}_{l,\pm1}\right]$$

$$\sigma_{sc} = \frac{\pi}{2k^2}\sum_l (2l+1)\left[\underbrace{|\alpha(l)|^2}_{\sim a_E} + \underbrace{|\beta(l)|^2}_{\sim a_M}\right]$$

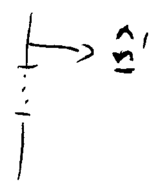
$$\sigma_{abs} = \frac{\pi}{2k^2}\sum_l (2l+1)\left[2 - |\alpha(l)+1|^2 - |\beta(l)+1|^2\right]$$

$$\sigma_{tot} = \sigma_{sc} + \sigma_{abs}$$

$$\frac{d\sigma_{sc}}{d\Omega} = \frac{\pi}{2k^2}\left|\sum_l \sqrt{2l+1}\left(\alpha_\pm(l)\,\underline{X}_{l,\pm1} \pm i\beta_\pm(l)\,\hat{\underline{n}}\times\underline{X}_{l,\pm1}\right)\right|^2$$

a_l, β_l from B/C on surface of the scatterer, e.g. $\underline{E}_{tan} = Z_s\,\hat{\underline{n}}\times\underline{B}/\mu_0$

<u>Diffraction:</u>



Scalar theory:

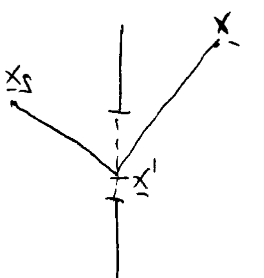
$$\psi(\underline{x}) = -\frac{1}{4\pi}\int_{S_1}\frac{e^{ikR}}{R}\left\{\underline{\nabla}'\psi(\underline{x}') + ik\left(1+\frac{i}{kR}\right)\frac{\underline{R}}{R}\psi(\underline{x}')\right\}\cdot\underline{n}'\,da'$$

Kirchhoff approx.: take $\psi(\underline{x}')$ and $\frac{\partial\psi}{\partial n'}(\underline{x}')$ of screen-less wave, $\int_{S_1}$ over apertures only. Can use for any (curved) screen.

<u>Modifications:</u> Dirichlet: use only $\psi(\underline{x}')$, double 2nd term, drop 1st term

Neumann: " " $\frac{\partial}{\partial n'}\psi(\underline{x}')$, " 1st " , " 2nd "

(for flat screen)



easy to interpret with Huygens' principle

$$\psi(\underline{x}) = \frac{k}{2\pi i}\int \frac{e^{ik|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|}\,\frac{e^{ik|\underline{x}-\underline{x}_s|}}{|\underline{x}_s-\underline{x}'|}\,\theta\,da'$$

$$\theta = \begin{cases}\frac{1}{2}(\cos\vartheta + \cos\vartheta_s), & K\\ \cos\vartheta, & D\\ \cos\vartheta_s, & N\end{cases}$$