

## Physics 506: Solutions to Assignment #5

### Problem 10.3

(a) Since  $\lambda \gg R$ , the fields are essentially constant over the size of the sphere. Furthermore the sphere can be treated as a perfectly conducting sphere since  $\delta \ll R$ . Therefore, to the  $0^{th}$  order, the problem can be approximated as a perfect conducting sphere in a static field. Thus the tangential component of the electric field vanishes ( $\vec{E}_{||} = 0$ ). To apply considerations of Sec. 8.1 to the power absorbed, we need to know the tangential component of the magnetic field  $\vec{H}_{||}$ . Consequently we need to solve the magnetostatic problem in field  $\vec{H} = \vec{H}_0 e^{-i\omega t}$  of the plane wave. *Note that we can always project the field of an unpolarized beam into two independent polarizations.* In spherical coordinates with  $\vec{H}$  pointing to the  $+z$  axis, the magnetic scalar potential has the form:

$$\Phi_M = -H_0 r \cos \theta + \frac{m}{4\pi r^2} \cos \theta$$

Here we have chosen the center-of-sphere as the coordinate origin. The first term is due to the uniform external field while the second term is due to the included magnetic dipole moment  $\vec{m}$  of the sphere. Note that  $\vec{m}$  and  $\vec{H}$  are in the same direction. Thus, the components of the magnetic field  $\vec{H} = -\nabla\Phi_M$  are:

$$H_r = -\frac{\partial\Phi_M}{\partial r} = H_0 \cos \theta + \frac{m}{2\pi r^3} \cos \theta$$

$$H_\theta = -\frac{1}{r} \frac{\partial\Phi_M}{\partial \theta} = -H_0 \sin \theta + \frac{m}{4\pi r^3} \sin \theta$$

and  $H_\phi = 0$ . For a perfect conductor,  $H_r = 0$  on the surface. Then,

$$H_r(r = R) = H_0 \cos \theta + \frac{m}{2\pi R^3} \cos \theta = 0 \quad \Rightarrow \quad m = -2\pi R^3 H_0 \quad \Rightarrow \quad \vec{m} = -2\pi R^3 \vec{H}_0$$

The  $0^{th}$  order magnetic field on the surface is thus

$$H_{||} = H_\theta = -\frac{3}{2} H_0 \sin \theta$$

(b) Since  $\delta \ll R$ , the power absorbed per unit area of surface is given by Eq. (8.15):

$$\frac{dP_{\text{loss}}}{da} = \frac{1}{2\sigma\delta} |\vec{K}_{\text{eff}}|^2 = \frac{1}{2\sigma\delta} |\vec{n} \times \vec{H}_{||}|^2 = \frac{9}{8\sigma\delta} |H_0|^2 \sin^2 \theta$$

The total power absorbed is

$$P_{\text{abs}} = \int \frac{dP_{\text{abs}}}{da} R^2 d\Omega = \frac{3\pi R^2}{\sigma\delta} |H_0|^2$$

Now note that the incident flux

$$I = \langle \vec{S} \cdot \vec{n} \rangle = \frac{1}{2} \vec{n} \cdot (\vec{E} \times \vec{H}^*) = \frac{1}{2} \vec{n} \cdot \left\{ (Z_0 \vec{H} \times \vec{n}) \times \vec{H}^* \right\} = \frac{Z_0}{2} |\vec{H}|^2 = \frac{1}{2} Z_0 |H_0|^2$$

the absorption cross section is

$$\sigma_{\text{abs}} = \frac{P_{\text{abs}}}{I} = \frac{6\pi R^2}{\sigma\delta Z_0}$$

Furthermore,

$$\delta = \sqrt{\frac{2}{\mu_0 \omega \sigma}} \quad \Rightarrow \quad \sigma = 6\pi R^2 \sqrt{\frac{\epsilon_0 \omega}{2\sigma}}$$

Therefore  $\sigma_{\text{abs}}$  is proportional to  $\sqrt{\omega}$  if the conductivity  $\sigma$  is independent of frequency.

**Problem 10.9(a)**

Useful integral

$$\int_0^\infty \frac{j_1^2(z)}{z} dz = \frac{\pi}{2} \int_0^\infty \frac{J_{3/2}^2(z)}{z^2} dz = \frac{1}{4}$$

Starting with the ‘Born’ approximation formula Eq. (10.31)

$$\frac{\vec{\epsilon}^* \cdot \vec{A}_{\text{sc}}^{(1)}}{D_0} = \frac{k^2}{4\pi} \int_{r \leq a} (\epsilon_r - 1) \vec{\epsilon}^* \cdot \vec{\epsilon}_0 e^{i\vec{q} \cdot \vec{r}} d\tau = \frac{(\epsilon_r - 1)}{4\pi} k^2 \vec{\epsilon}^* \cdot \vec{\epsilon}_0 \int_{r \leq a} e^{i\vec{q} \cdot \vec{r}} d\tau$$

Define

$$\begin{aligned} I &= \frac{1}{4\pi} \int e^{i\vec{q} \cdot \vec{r}} d\tau = \frac{1}{4\pi} \int_0^a r^2 dr \int d\Omega e^{iqr \cos \theta} = \int_0^a r^2 dr \frac{1}{2} \int_{-1}^{+1} e^{iqr \cos \theta} d(\cos \theta) \\ &= \int_0^a r^2 \left\{ \frac{\sin(qr)}{qr} \right\} dr = \frac{1}{q^3} \int_0^{qa} \xi \sin \xi d\xi = \frac{1}{q^3} (\sin(qa) - qa \cos(qa)) = a^3 \frac{j_1(qa)}{qa} \end{aligned}$$

The differential scattering cross section, averaged over initial polarizations and summed over final polarizations, is

$$\frac{d\sigma}{d\Omega} = (ka)^4 a^2 |\epsilon_r - 1|^2 \left| \frac{j_1(qa)}{qa} \right|^2 \cdot \left\{ \frac{1}{2} \sum_{\text{pol.}} |\vec{\epsilon}^* \cdot \vec{\epsilon}_0|^2 \right\}$$

where the summation is over initial and final state polarizations:

$$\frac{1}{2} \sum_{\text{pol.}} |\vec{\epsilon}^* \cdot \vec{\epsilon}_0|^2 = \frac{1}{2} (1 + \cos^2 \theta)$$

Now note

$$q^2 = k^2 |\vec{n}_0 - \vec{n}|^2 = k^2 (2 - 2\vec{n}_0 \cdot \vec{n}) = 2k^2 (1 - \cos \theta)$$

For large  $ka \gg 1$ ,  $qa = ka \sqrt{2(1 - \cos \theta)}$  can be large compared to unity. Since

$$\left| \frac{j_1(qa)}{qa} \right|^2 = \mathcal{O} \left\{ \frac{1}{(qa)^4} \right\}$$

for large  $qa$ , the scattering is mainly confined to small  $qa$ . Small  $qa$  and  $ka \gg 1$  imply  $\theta \ll 1$ , *i.e.* the differential cross section is sharply peaked in the forward direction.

$$d\Omega = d\phi d(\cos \theta) = d\phi \frac{1}{2k^2} d(q^2) = \frac{1}{2k^2 a^2} d(q^2 a^2) d\phi$$

Let  $z = qa$ , then

$$d\Omega = \frac{1}{2k^2 a^2} (2z dz) d\phi = \frac{1}{(ka)^2} z dz$$

The total scattering cross section can be written

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{(ka)^4 a^2 |\epsilon_r - 1|^2}{(ka)^2} \int_0^{2\pi} d\phi \int_0^{2ka} z dz \left| \frac{j_1(z)}{z} \right|^2 \cdot \frac{1}{2} (1 + \cos^2 \theta)$$

Since  $\theta \ll 1$ ,  $(1 + \cos^2 \theta)/2 \rightarrow 1$  and  $(qa)_{\text{max}} = z_{\text{max}} \rightarrow \infty$ , then

$$\sigma \approx (ka)^2 |\epsilon_r - 1|^2 a^2 \times 2\pi \int_0^\infty \frac{j_1^2(z)}{z} dz = \frac{\pi}{2} (ka)^2 |\epsilon_r - 1|^2 a^2$$

**Problem 10.12**

(a) The diffracted field due to a plane surface is given by Eq. (10.101):

$$\vec{E}(\vec{r}) = \frac{1}{2\pi} \nabla \times \int_{\text{apertures}} \left\{ \vec{n} \times \vec{E}(\vec{r}') \right\} \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} da'$$

Here  $\vec{n}$  is a unit normal. In radiation zone,  $|\vec{r}-\vec{r}'| \approx r - \vec{n} \cdot \vec{r}'$ . Thus

$$\vec{E}(\vec{r}) \approx \frac{1}{2\pi} \nabla \times \left\{ \frac{e^{ikr}}{r} \int_{\text{aperture}} (\vec{n} \times \vec{E}(\vec{r}')) e^{-i\vec{k} \cdot \vec{r}'} da' \right\} \approx \frac{i}{2\pi} \frac{e^{ikr}}{r} \vec{k} \times \left\{ \int_{\text{aperture}} (\vec{n} \times \vec{E}(\vec{r}')) e^{-i\vec{k} \cdot \vec{r}'} da' \right\}$$

Choose a rectangular coordinate system with the  $x-z$  plane as the plane of incidence and its origin at the center of the aperture, thus  $\vec{n} = \hat{z}$  and the incident wave vector  $\vec{k}_0 = k(\cos \alpha \hat{z} + \sin \alpha \hat{x})$ . Let  $(\theta, \phi)$  be the spherical angles of the outgoing wave vector  $\vec{k}$  and  $(\rho', \beta', 0)$  be the polar coordinates of  $\vec{r}'$ , thus

$$\vec{k} = k(\sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}), \quad \vec{r}' = \rho' \cos \beta' \hat{x} + \rho' \sin \beta' \hat{y}$$

Consequently

$$\vec{k} \cdot \vec{r}' = k \rho' \sin \theta (\cos \phi \cos \beta' + \sin \phi \sin \beta') = k \sin \theta \rho' \cos(\phi - \beta'), \quad \text{and} \quad \vec{k}_0 \cdot \vec{r}' = k \sin \alpha \rho' \cos \beta'$$

Therefore, the field

$$\begin{aligned} \vec{E}(\vec{r}) &= \frac{i}{2\pi} \frac{e^{ikr}}{r} \vec{k} \times \int_{\text{aperture}} \vec{n} \times \left\{ E_0 e^{i\vec{k}_0 \cdot \vec{r}'} \hat{y} \right\} e^{-i\vec{k} \cdot \vec{r}'} da' \\ &= \frac{i}{2\pi} \frac{e^{ikr}}{r} \vec{k} \times \int_{\text{aperture}} \vec{z} \times \left\{ E_0 e^{ik \sin \alpha \rho' \cos \beta'} \hat{y} \right\} e^{-ik \sin \theta \rho' \cos(\phi - \beta')} \rho' d\rho' d\beta' \\ &= -\frac{i}{2\pi} E_0 \frac{e^{ikr}}{r} \vec{k} \times \hat{x} \int_0^a \rho' d\rho' \int_0^{2\pi} e^{ik \rho' (\sin \alpha \cos \beta' - \sin \theta \cos(\phi - \beta'))} d\beta' \end{aligned}$$

Now note the exponent of the integrand

$$\begin{aligned} \sin \alpha \cos \beta - \sin \theta \cos(\phi - \beta) &= \sin \alpha \cos \beta - \sin \theta \cos \phi \cos \beta - \sin \theta \sin \phi \sin \beta \\ &= -\{(\sin \theta \cos \phi - \sin \alpha) \cos \beta + (\sin \theta \sin \phi) \sin \beta\} \\ &= -\xi \left( \frac{\sin \theta \cos \phi - \sin \alpha}{\xi} \cos \beta + \frac{\sin \theta \sin \phi}{\xi} \sin \beta \right) \\ &= -\xi (\cos \beta \cos \beta_0 + \sin \beta \sin \beta_0) = -\xi \cos(\beta - \beta_0) \end{aligned}$$

Here

$$\xi^2 = (\sin \theta \cos \phi - \sin \alpha)^2 + (\sin \theta \sin \phi)^2 = \sin^2 \theta + \sin^2 \alpha - 2 \sin \alpha \sin \theta \cos \phi$$

and

$$\cos \beta_0 = \frac{\sin \theta \cos \phi - \sin \alpha}{\xi}; \quad \sin \beta_0 = \frac{\sin \theta \sin \phi}{\xi}$$

Applying the integral

$$\frac{1}{2\pi} \int_0^{2\pi} e^{\pm ix \sin \theta} d\theta = J_0(x)$$

we get the electric field

$$\begin{aligned}
\vec{E}(\vec{r}) &= -\frac{iE_0}{2\pi} \frac{e^{ikr}}{r} \vec{k} \times \hat{x} \int_0^a \rho' d\rho' \int_0^{2\pi} e^{-ik\rho'\xi \cos(\beta' - \beta_0)} d\beta' \\
&= -iE_0 \frac{e^{ikr}}{r} \vec{k} \times \hat{x} \int_0^a \rho' J_0(k\rho'\xi) d\rho' \\
&= -ia^2 E_0 \frac{e^{ikr}}{r} \vec{k} \times \hat{x} \frac{J_1(ka\xi)}{ka\xi} \equiv \mathcal{A}(\vec{k} \times \hat{x})
\end{aligned}$$

Now that

$$\vec{H} = \frac{1}{Z_0} \hat{r} \times \vec{E} = \frac{\mathcal{A}}{Z_0} \hat{r} \times (\vec{k} \times \hat{x})$$

The average Poynting vector

$$\langle \vec{S} \rangle = \frac{1}{2} \text{Re} \left\{ \vec{E} \times \vec{H}^* \right\} = \frac{1}{2Z_0} |\vec{E}|^2 \hat{r} = \frac{1}{2Z_0} |\mathcal{A}|^2 \cdot |\vec{k} \times \hat{x}|^2 \hat{r}$$

Note that  $|\vec{k} \times \hat{x}|^2 = k^2(\sin^2 \theta \sin^2 \phi + \cos^2 \theta)$ . The time-averaged diffracted power per unit solid angle

$$\frac{dP}{d\Omega} = r^2 \langle \vec{S} \rangle \cdot \hat{r} = \frac{r^2}{2\mu_0 c} |\mathcal{A}|^2 \cdot |\vec{k} \times \hat{x}|^2 = \frac{1}{2Z_0} k^2 a^4 |E_0|^2 \left| \frac{J_1(ka\xi)}{ka\xi} \right|^2 (\sin^2 \theta \sin^2 \phi + \cos^2 \theta)$$

The explicit dependence on  $\phi$  of the differential power is the result of the polarization of the incoming wave. Now that the incident power  $P_i$ :

$$P_i = \int \frac{1}{2} (\vec{E} \times \vec{H}^*) \cdot \vec{n}_0 da' = \frac{\pi a^2}{2Z_0} |E_0|^2 \cos \alpha \quad \Rightarrow \quad |E_0|^2 = \frac{2Z_0}{\pi a^2 \cos \alpha} P_i$$

Thus

$$\left( \frac{dP}{d\Omega} \right)_\perp = \frac{P_i}{\cos \alpha} \frac{(ka)^2}{4\pi} (\sin^2 \theta \sin^2 \phi + \cos^2 \theta) \left| \frac{2J_1(ka\xi)}{ka\xi} \right|^2$$

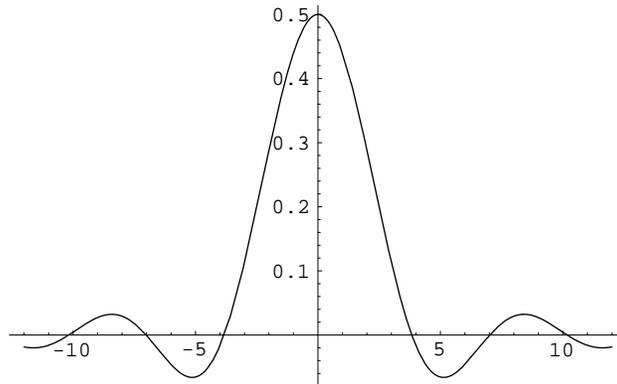
(b) For the polarization in the plane of incidence, the differential diffractive power is given by Eq. (10.114):

$$\left( \frac{dP}{d\Omega} \right)_\parallel = P_i \cos \alpha \frac{(ka)^2}{4\pi} (\sin^2 \theta \cos^2 \phi + \cos^2 \theta) \left| \frac{2J_1(ka\xi)}{ka\xi} \right|^2$$

For normal incidence ( $\alpha = 0$ ), the  $\perp$  case with  $\phi = 0$  and  $\parallel$  case with  $\phi = \pi/2$  should be identical. Indeed the two formula are indeed the same. Furthermore for  $\alpha = 0$ , the diffractive power for an unpolarized beam is

$$\begin{aligned}
\frac{dP}{d\Omega} &= \frac{1}{2} \left\{ \left( \frac{dP}{d\Omega} \right)_\perp + \left( \frac{dP}{d\Omega} \right)_\parallel \right\} \\
&= P_i \frac{(ka)^2}{4\pi} \frac{1}{2} (1 + \cos^2 \theta) \left| \frac{2J_1(ka\xi)}{ka\xi} \right|^2
\end{aligned}$$

As expected, the diffraction pattern of an unpolarized beam is independent of the azimuthal angle and is determined by the function  $J_1(x)/x$ , which is plotted below.



The vector results above are very similar to

$$\frac{dP}{d\Omega} = \frac{P_i}{\cos \alpha} \frac{(ka)^2}{4\pi} \left( \frac{\cos \alpha + \cos \theta}{2} \right)^2 \left| \frac{2J_1(ka\xi)}{ka\xi} \right|^2$$

of the scalar Kirchhoff approximation apart from the angular factor resulting from polarization.

**Problem 10.18**

(a) In the long wavelength limit, the small circular hole can be viewed as electric and magnetic dipoles with moments

$$\vec{p}_{\text{eff.}} = \frac{4\epsilon_0}{3} a^3 \vec{E}_0; \quad \vec{m}_{\text{eff.}} = -\frac{8}{3} a^3 \vec{H}_0$$

Therefore, the diffracted electric field in the Fraunhofer zone is given by Eq. (10.2):

$$\vec{E}(\vec{r}) = \frac{k^2}{4\pi\epsilon_0} \frac{e^{ikr}}{r} (\vec{n} \times \vec{p}_{\text{eff.}}) \times \vec{n} - \frac{k^2}{4\pi\epsilon_0} \frac{e^{ikr}}{r} (\vec{n} \times \frac{\vec{m}_{\text{eff.}}}{c})$$

where  $\vec{n} = \vec{k}/k$ . Inserting the effective dipole moments, we have

$$\begin{aligned} \vec{E}(\vec{r}) &= \frac{k^2}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \frac{4\epsilon_0 a^3}{3} (\vec{n} \times \vec{E}_0) \times \vec{n} + \frac{k^2}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \frac{8a^3}{3c} (\vec{n} \times \vec{H}_0) \\ &= \frac{k^2 a^3}{4\pi\epsilon_0} \frac{e^{ikr}}{r} (\vec{n} \times \vec{E}_0) \times \vec{n} + \frac{2k^2 a^3}{3\pi} \frac{e^{ikr}}{r} (\vec{n} \times c\vec{B}_0) \\ &= \frac{k^2 a^3}{3\pi} \frac{e^{ikr}}{r} \left\{ \vec{n} \times (\vec{E}_0 \times \vec{n}) + 2c\vec{n} \times \vec{B}_0 \right\} \end{aligned}$$

With explicit time-dependence, the field can be written as

$$\vec{E} = \frac{e^{ikr - i\omega t}}{3\pi r} k^2 a^3 \left\{ 2c \frac{\vec{k}}{k} \times \vec{B}_0 + \frac{\vec{k}}{k} \times (\vec{E}_0 \times \frac{\vec{k}}{k}) \right\}$$

(b) Choose a coordinate system such that  $\vec{E}_0$  is along the  $z$ -axis,  $\vec{B}_0$  is along the  $x$ -axis and let  $\vec{k} = k(\hat{x} \sin \theta \cos \phi + \hat{y} \sin \theta \sin \phi + \hat{z} \cos \theta)$ , the time-averaged radiation power per unit solid angle is

$$\frac{dP}{d\Omega} = \frac{r^2}{2Z_0} |\vec{E}|^2 = \frac{1}{18\pi^2 Z_0} k^4 a^6 \left| 2c \frac{\vec{k}}{k} \times \vec{B}_0 + \frac{\vec{k}}{k} \times (\vec{E}_0 \times \frac{\vec{k}}{k}) \right|^2$$

Note that

$$\begin{aligned} |2c\vec{n} \times \vec{B}_0 + \vec{n} \times (\vec{E}_0 \times \vec{n})|^2 &= |2c\vec{n} \times \vec{B}_0 + \vec{E}_0 - (\vec{n} \cdot \vec{E}_0)\vec{n}|^2 \\ &= 4c^2 |\vec{n} \times \vec{B}_0|^2 + 4c \text{Re} \left\{ (\vec{n} \times \vec{B}_0) \times \vec{E}_0^* \right\} + |\vec{E}_0|^2 - |\vec{n} \cdot \vec{E}_0|^2 \\ &= 4c^2 |B_0|^2 (\sin^2 \theta \sin^2 \phi + \cos^2 \theta) - 4c \sin \theta \sin \phi \text{Re}(\vec{E}_0^* \cdot \vec{B}_0) + |E_0|^2 \sin^2 \theta \end{aligned}$$

Thus, the differential power

$$\frac{dP}{d\Omega} = \frac{1}{18\pi^2 Z_0} k^4 a^6 \left\{ 4c^2 |B_0|^2 (\sin^2 \theta \sin^2 \phi + \cos^2 \theta) - 4c \sin \theta \sin \phi \text{Re}(\vec{E}_0^* \cdot \vec{B}_0) + |E_0|^2 \sin^2 \theta \right\}$$

The total power transmitted

$$\begin{aligned} P &= \int \frac{dP}{d\Omega} d\Omega = \frac{1}{18\pi^2 Z_0} k^4 a^6 \int_0^{\pi/2} \sin \theta d\theta \int_0^{2\pi} d\phi \left\{ 4c^2 |B_0|^2 (\sin^2 \theta \sin^2 \phi + \cos^2 \theta) - 4c \text{Re}(\vec{E}_0^* \cdot \vec{B}_0) \sin \theta \sin \phi + |E_0|^2 \sin^2 \theta \right\} \\ &= \frac{2}{27\pi Z_0} k^4 a^6 (4c^2 |B_0|^2 + |E_0|^2) \end{aligned}$$