Outline

- Elements of non-cooperative game theory.
  - Preference, utility, actions, strategies.
- Normal Form games.
  - Identifying Nash Equilibria
  - Domination
  - Mixed Strategies
- The Growing Family of Equilibrium Concepts

Components of a Game

- players
- actions
- strategies
- information
- outcomes
- payoffs
- Equilibrium concept

Game Theory Fundamentals

- Player goals are represented by utility functions with utility defined over outcomes.

- Actions and Strategies
  - A strategy is a plan of action.
  - In games that can be modeled as if they are simultaneous, actions and strategies are equivalent.
  - In other games, strategies and actions are quite different with strategies being the primary choice of interest.

- The combination of actions by all players determines a payoff for each player.

A normal form game

- By convention, the payoff to the so-called row player is the first payoff given, followed by the payoff to the column player.

Graduate School

	Study	**Loaf**
Study	100,100	50,0
Loaf	0, 50	-10, -10

Practical Description

- The normal form representation of a game specifies:
  - The players in the game.
  - The strategies available to each player.
  - The payoff received by each player for each combination of strategies that could be chosen by the players.

- Actions are modeled as if they are chosen simultaneously.
  - The players need not really choose simultaneously, it is sufficient that they act without knowing each others' choices.

Components of a Normal Form Game

• Players	A small number.
• Actions	Define columns and rows.
• Strategies.	Define columns and rows.
• Information.	Complete.
• Outcomes.	Represented by vectors in cells.
• Payoffs.	Elements of the vectors.
• Equilibrium concept.	Nash.

Technical Definition

- 1 to n: players in an n-player game.
- S_i: player i's strategy set.
- S_i: an arbitrary element of S_i.
- $u_i(s_i)$: player i's payoff function.

- Definition: The normal-form representation of an n-player game specifies the players' strategy spaces $S_1,\ldots,S_n$ and their payoff functions $u_1,\ldots,u_n$.
- We denote the game by $G=\{S_1,\ldots S_n;u_1,\ldots u_n\}$.

Nash Equilibrium

- For an equilibrium prediction to be correct, it is necessary that each player be willing to choose the strategy described in the equilibrium.
- Equilibrium represents the outcome of mutual and joint adaptation to shared circumstances.
- If the theory offers strategies that are not a Nash equilibrium, then at least one player will have an incentive to deviate from the theory's prediction, so the theory will be falsified by the actual play of the game.

Technical Definition

- In the n-player normal-form game $G=\{S_1,\ldots S_n; u_1,\ldots u_n\}$, the strategies $(s_1^*,\ldots s_n^*)$ are a Nash equilibrium if, for each player i,
  - s^*_i is (at least tied for) player i's best response
  - to the strategies specified for the n-1 other players, $(s^*_1,\ldots s^*_{i-1}, s^*_{i+1},\ldots s^*_n)$: $u_i(s^*_1,\ldots s^*_{i-1}, s^*_i, s^*_{i+1},\ldots s^*_n) \geq u_i(s^*_1,\ldots s^*_{i-1}, s_i, s^*_{i+1},\ldots s^*_n)$
  - for every feasible strategy s_i in S_i;
  - that is, s^*_i solves $\max_{s_i \in S_i} u_i(s^*_1,\ldots s^*_{i-1}, s_i, s^*_{i+1},\ldots s^*_n)$.

- If the situation is modeled accurately, NE represent social outcomes that are self-enforcing.

- Any outcome that is not a NE can be accomplished only by application of an external mechanism.

Ways to identify NE – in order of ease.

- *Rule:* As the level of conflict increases, so does the work required to derive a solution.
  - Identify pairs of dominant strategies.
  - Eliminate dominated strategies.
  - Identify stable pairs of pure strategies.
  - Identify stable pairs of mixed (probabilistic) strategies.

Strictly dominated

- In the normal-form game $G=\{S_1,\ldots S_n;\ u_1,\ldots u_n\}$, let s'_i and s''_i be feasible strategies for player i (i.e., s'_i and s''_i are members of S_i).

- Strategy s'_i is **strictly dominated by** strategy s''_i if
  - for each feasible combination of the other players' strategies,
  - i's payoff from playing s'_i is strictly less than i's payoff from playing s''_i: $u_i(s_1,\ldots s_{i-1},\ s'_i,\ s_{i+1},\ldots s_n) < u_i(s_1,\ldots s_{i-1},\ s''_i,\ s_{i+1},\ldots s_n)$ for each $(s_1,\ldots s_{i-1},\ s_{i+1},\ldots s_n)$ that can be constructed from the other players; strategy spaces $S_1,\ldots S_{i-1},\ S_{i+1},\ldots S_n$.

Elimination of dominated strategies

	Left	**Middle**	**Right**
Up	1,0	1,2	0,1
Down	0,3	0,1	2,0

Figure 1.1.1. Iterated domination produces a solution.

	Left	**Middle**	**Right**
Top	0,4	4,0	5,3
Middle	4,0	0,4	5,3
Bottom	3,5	3,5	6,6

Figure 1.1.4. Iterated elimination produces no solution.

Requirements for Iterated Domination

- If we want to be able to apply the process for an arbitrary number of steps, we need to assume that it is *common knowledge* that the players are rational.
- We need to assume not only that all the players are rational, but also that all the players know that all the players are rational, and that all the players know that all the players know that all the players are rational, and so on, *ad infinitum*.
- In the many cases where there is no or few strictly dominated strategies, the process produces very imprecise predictions.

Example 1: A game with a dominated strategy.

	Left	**Right**
Top	8, 10	-100, 9
Bottom	7, 6	6, 5

Example 2: A more complicated game: with dominated strategies.

	Left	**Middle**	**Right**
Top	4, 3	5, 1	6, 2
Middle	2, 1	8, 4	3, 6
Bottom	3, 0	9, 6	2, 8

NE: Fun facts

- If iterated elimination of dominated strategies eliminates all but one strategy for each player, then these strategies are the unique NE.
- There can be strategies that survive iterated elimination of strictly dominated strategies but are not part of any Nash equilibrium.
- If most models are to produce a unique solution, the solution must be a Nash equilibrium.
- A game can have multiple Nash equilibria. The precision of its predictive power at such moments lessens.

	Left	**Middle**	**Right**
Top	0,4	4,0	5,3
Middle	4,0	0,4	5,3
Bottom	3,5	3,5	6,6

Figure 1.1.5. Iterated elimination produced no solution. Find the Nash Equilibrium.

	Opera	**Fight**
Opera	2,1	0,0
Fight	0,0	1,2

Battle of the Sexes

Solving for MS-NE

- Row chooses “top” with probability p and bottom with probability 1-p.
- Column chooses “left” with probability q and “right” with probability 1-q.

	Left	**Right**
Top	4, -4	1, -1
Bottom	2, -2	3, -3

- Players choose strategies to make the other indifferent.
  - $4q+1(1-q)=2q+3(1-q)$
  - $-4p-2(1-p)=-1p-3(1-p)$

- The MS-NE is: p=.25, q=.5.
  - The expected value of either Row strategy is 2.5 and of either Column strategy is –2.5

Mixed strategy NE

- A mixed strategy Nash Equilibrium does not rely on an player flipping coins, rolling, dice or otherwise choosing a strategy at random.

- Rather, we interpret player j's mixed strategy as a statement of player i's uncertainty about player j's choice of a pure strategy.

- In games of pure conflict, where there is no pure strategy Nash equilibria, the mixed strategy equilibriums are chosen in a way to make the other player indifferent between all of their mixed strategies.
  - To do otherwise is to give others the ability to benefit at your expense. Information provided to another player that makes them better off makes you worse off.

Mixed Strategies

- In the normal-form game $G=\{S_1,\ldots S_n; u_1,\ldots u_n\}$, suppose $S_i = \{s_{i1},\ldots s_{iK}\}$. Then a **mixed strategy** for player i is a probability distribution $p_i=(p_{i1},\ldots p_{ik})$, where $0 \leq p_{ik} \leq 1$ for $k=1,\ldots,K$ and $p_{i1}+\ldots+p_{iK}=1$.

	Left	**Right**
Top	3, -	0, -
Middle	0, -	3, -
Bottom	1, -	1, -

Figure 1.3.1. Bottom is strictly dominated by a mixed strategy.

	Left	**Right**
Top	3, -	0, -
Middle	0, -	3, -
Bottom	2, -	2, -

Figure 1.3.2. Bottom is a best response to mixed strategies by the column player in which 1/3 < q < 2/3.

Technical Definition

- Let $v_i(p_i, p_{-i})$ be the expected payoff of mixed strategy p_i to player i given that the other player chooses mixed strategy p_{-i}.

- Then, in the two player normal-form game $G=\{S_1, \ldots S_n; u_1, \ldots u_n\}$, the mixed strategies $(p_1^*, \ldots p_n^*)$ are a Nash equilibrium if each player's mixed strategy is
  - a best response to the other player's mixed strategy:
  - $v_1(p^*_1, p^*_2) \geq v_1(p_1, p^*_2)$ for every probability distribution p_1 over S_1 and
  - $v_2(p^*_1, p^*_2) \geq v_2(p^*_1, p_2)$ for every probability distribution p_2 over S_2.

	Halves	More for Me
Halves	150, 150	0, 0
More for you	125, 175	100, 200

	Left	Right
Top	4, -4	1, -1
Bottom	2, -2	3, -3

	Left	Right
Top	4, 1	1, 2
Bottom	2, 3	3, 6

The last word.

- Theorem (Nash 1950): In the n-player normal-form game $G=\{S_1, \ldots S_n; u_1, \ldots u_n\}$, if n is finite and S_i is finite for every i then there exists at least one Nash Equilibrium, possibly involving mixed strategies.

Equilibrium Concepts

Move sequence:	static		dynamic	
Information:	complete	incomplete	complete	incomplete
Appropriate Nash Equilibrium concept	**Generic**	Bayesian	Subgame perfect	Perfect Bayesian, sequential

- What is the set of self enforcing best responses?
- The equilibrium concepts build upon those of simpler games.
- Each subsequent concept, while more complex, also allows more precise conclusions from increasingly complex situations