Dynamic Bayesian Games

Everyone is aware of the informational uncertainties of others and thinks about implications for the past, present and future

Perfect Concepts

- The subgame perfect equilibrium concept adds to the Nash concept the requirement that players choose optimally in subgames.
- But, a proper subgame cannot begin at an information set containing multiple nodes.
- Perfect Bayesian equilibrium adds to Nash the requirement that players choose optimally given their beliefs about the rest of the game.

Sequential Rationality

- A pair of beliefs and strategies is **sequentially rational** iff from each information set, the moving player's strategy maximizes its expected utility for the remainder of the game given its beliefs and all players' strategies.

- Sequential rationality allows a process akin to backwards induction on games of incomplete information.

Perfect Bayesian Equilibrium

- A perfect Bayesian equilibrium is a belief-strategy pairing such that
  - the strategies are sequentially rational given the beliefs
  - and the beliefs are calculated from the equilibrium strategies by Bayes' Theorem whenever possible.
- A defection from the equilibrium path does not increase the chance that others will play "irrationally."
- Every finite n-person game has at least one perfect Bayesian equilibrium in mixed strategies.

Requirements for PBE in Extensive-Form Games

- An information set is **on the equilibrium path** if it will be reached with positive probability ⇔ the game is played according to the equilibrium strategies.
- On the equilibrium path, Bayes' Rule and equilibrium strategies determine beliefs.
- *Off* the equilibrium path, Bayes' Rule and equilibrium strategies determine beliefs *where possible*.

Implications

- In a PBE, players cannot threaten to play strategies that are strictly dominated beginning at any information set off the equilibrium path.
- A single pass working backwards through the tree (typically) will not suffice to compute a PBE.

Do the game on Gibbons 181.

Look for Nash Equilibria, then
discuss the problem with the answer.

Do Exercise 6.10 in Morrow

Communication Games (Gibbons 174-5)

- A **signaling game** involves two players (one with private information, the other without) and two moves (first a signal sent by the informed player, then a response by the uninformed player).
  - The key idea is that communication can occur if one type of the informed player is willing to end a signal that would be too expensive for another type to send.

- **Cheap-talk game**: a signaling game in which all messages are free.
  - The extent of communication is determined by the commonality of the players' interests.

Signaling Game

- A dynamic game of incomplete information involving two players: a Sender (S) and a Receiver (R).
- Nature draws type t_i for S from $T=\{t_1, \ldots t_I\}$ according to distribution $p(t_i)$, where $p(t_i)>0 \ \forall i$ and $\sum_I p(t_i)=1$.
- S observes t_i, then sends message m_j from $M=\{m_1, \ldots, m_J\}$.
- R observes m_j (not t_i), and then chooses reaction a_k from $A=\{a_1, \ldots, a_K\}$.
- Payoffs: $U_S(t_i, m_j, a_k)$, $U_R(t_i, m_j, a_k)$.

Requirements for PS-PBE in Signaling Games

1. R must have a belief $\mu(t_i|m_j)$ about who sent message m_j, where $\mu(t_i|m_j) \geq 0$, $\forall t_i \in T$, $\sum_{ti \in T} \mu(t_i|m_j) = 1$

2. For each $m_j \in M$, $a^*(m_j)$ solves $\max_{ak \in A} \sum_{ti \in T} \mu(t_i|m_j) U_R(t_i, m_j, a_k)$.

- For each $t_i \in T$, $m^*(t_i)$ solves $\max_{mj \in M} U_S(t_i, m_j, a_k)$.

  – Belief-strategy pairs must be **sequentially rational:** from each information set, the moving player's strategy maximizes its expected utility for the remainder of the game given its beliefs and all players' strategies.

Requirements for PS- PBE in Signaling Games

3. For each $m_j \in M$, if $\exists\ t_i \in T$ such that $m^*(t_i)=m_j$, then R's belief must follow from Bayes' Rule and S's strategy.

$$\mu(t_i|m_j) = p(t_i)/\Sigma_{ti \in T} = p(t_i)$$

(stated for pure strategies).

- A pure-strategy PBE in a signaling game is a set of belief-strategy pairs $[m^*(t_i), a^*(m_j)\ \mu(t_i|m_j)]$ satisfying requirements 1-3.

Morrow, Table 7.1

Concept	Replies judged	Key comparison	Beliefs used?	Beliefs off the equ path?
Nash	Along the equ path	Complete Strategies	No	Irrelevant
Subgame Perfect	In proper subgames	Strategies within proper subgames	No	Irrelevant
Perfect Bayesian	At all information sets	Seq. Rationality at all Info sets	Yes	Can be chosen.
Perfect	At all information sets	Against trembles. No Weak. Dom. S.	No	Irrelevant

Do Gibbons 189

What to check for

- Given the receiver's response, is the signal utility maximizing for type 1?
- Given the receiver's response, is the signal utility maximizing for type 2?
- Given the sender's strategy, does the response to *L* maximize expected utility?
- Given the sender's strategy, does the response to *R* maximize expected utility?
- If a signal is off the equilibrium path, do there exist off-the-path beliefs that can sustain the equilibrium?

t_1	t_2	A_R|L	A_R |R	PBE?	t_1	t_2	A_R |L	A_R |R	PBE?
L	L	u	u		R	L	u	u	
L	L	u	d		R	L	u	d	
L	L	d	u		R	L	d	u	
L	L	d	d		R	L	d	d	
R	R	u	u		L	R	u	u	
R	R	u	d		L	R	u	d	
R	R	d	u		L	R	d	u	
R	R	d	d		L	R	d	d	

t_1	t_2	A|L	A|R	PBE?	t_1	t_2	A|L	A|R	PBE?
L	L	u	u	no	R	L	u	u	YES if $p=0$ & $q=1$
L	L	u	d	YES if $p=.5$ & $q \leq 2/3$	R	L	u	d	no
L	L	d	u	no	R	L	d	u	no
L	L	d	d	no	R	L	d	d	no
R	R	u	u	no	L	R	u	u	no
R	R	u	d	no	L	R	u	d	no
R	R	d	u	no	L	R	d	u	no
R	R	d	d	no	L	R	d	d	no

Ordeshook, Problem 7

Note: The class payoffs are independent of the TA's type. Therefore, the stated strategy must be part of any B-NE.

1. Is a B-N E.
2. Is not. Given the class response, type R should not place any weight on "Beach."
3. Is a B-NE.
4. Is not. Type R should choose "Office."

Ordeshook, Problem 10

1. Is not. If you vote YES, regardless of the signal, your expected utility is –7. Voting NO yields an expected utility of –3.
2. Is not. The signal HOUSE implies that the type is INS. Your best response is NO.
3. Is not.
   - Upon seeing HOUSE the posterior probability of INS is .85 (.35/.41).
     - The expected utility of YES: -8.5, NO:-1.5.
   - Upon seeing NO, the posterior probability of INS is .59 (.35/.59).
     - The expected utility of YES: -5.9, NO:-4.1.
   - Your best response is NO.
4. Is not. If you will vote YES given the signal NO, the campaigner's best response is HOUSE.
5. Same as 1.
6. Is not. If you choose NO regardless of the signal, the campaigner's best response is also NO.

Spence (1973)

- The seminal *signaling model* focuses on the plight of an employer.

- The employer prefers skilled workers.
- She cannot observe skills in advance.
- Skilled applicants can purchase “education” at a lower cost than others.
- She observes applicants’ education.
- Education persuades her of the applicant's skills only if it and skills correlate sufficiently in equilibrium.

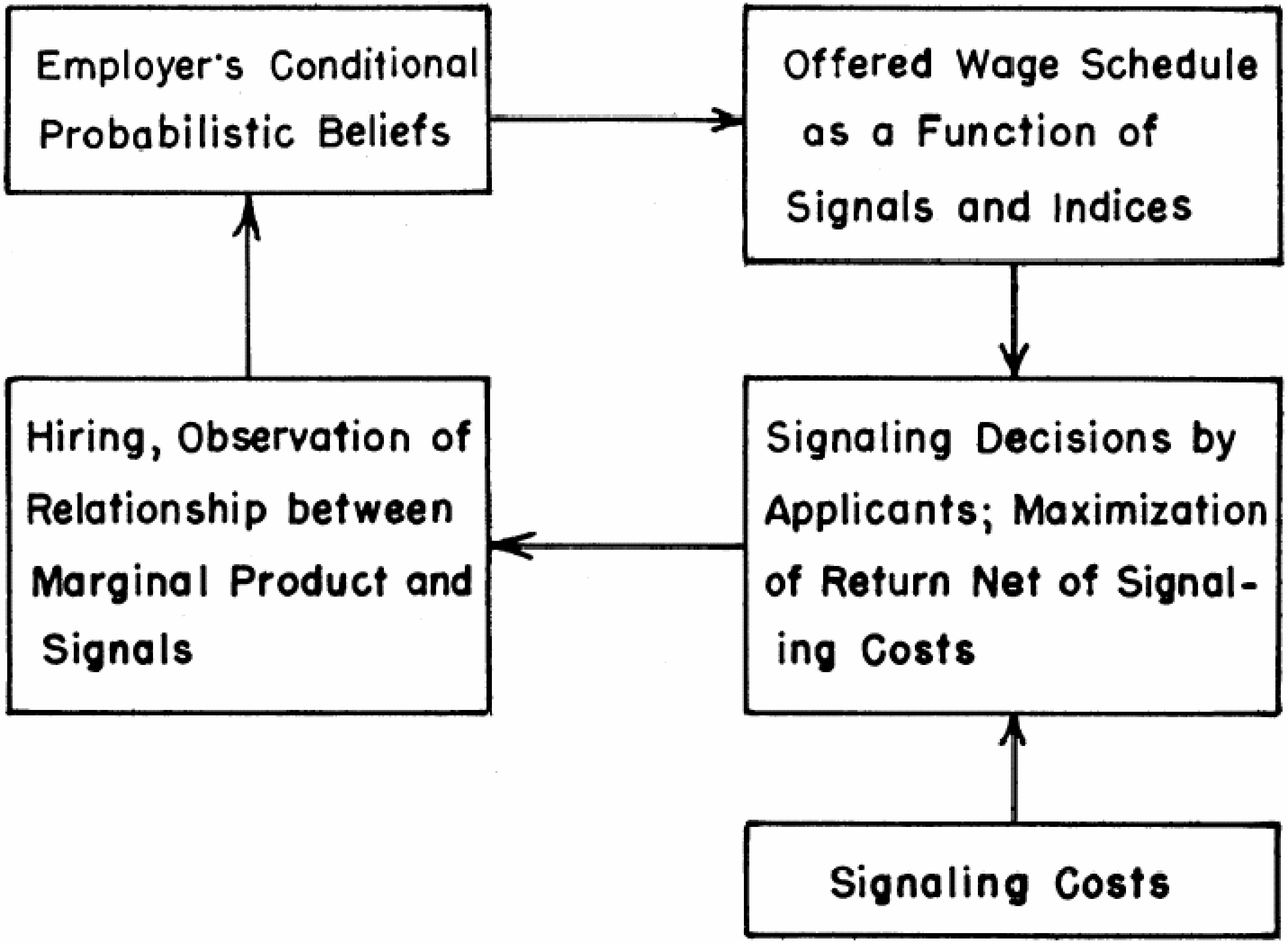

FIGURE I

Informational Feedback in the Job Market

TABLE I

DATA OF THE MODEL

Group	Marginal product	Proportion of population	Cost of education level y
I	1	q_1	y
II	2	$1-q_1$	$y/2$

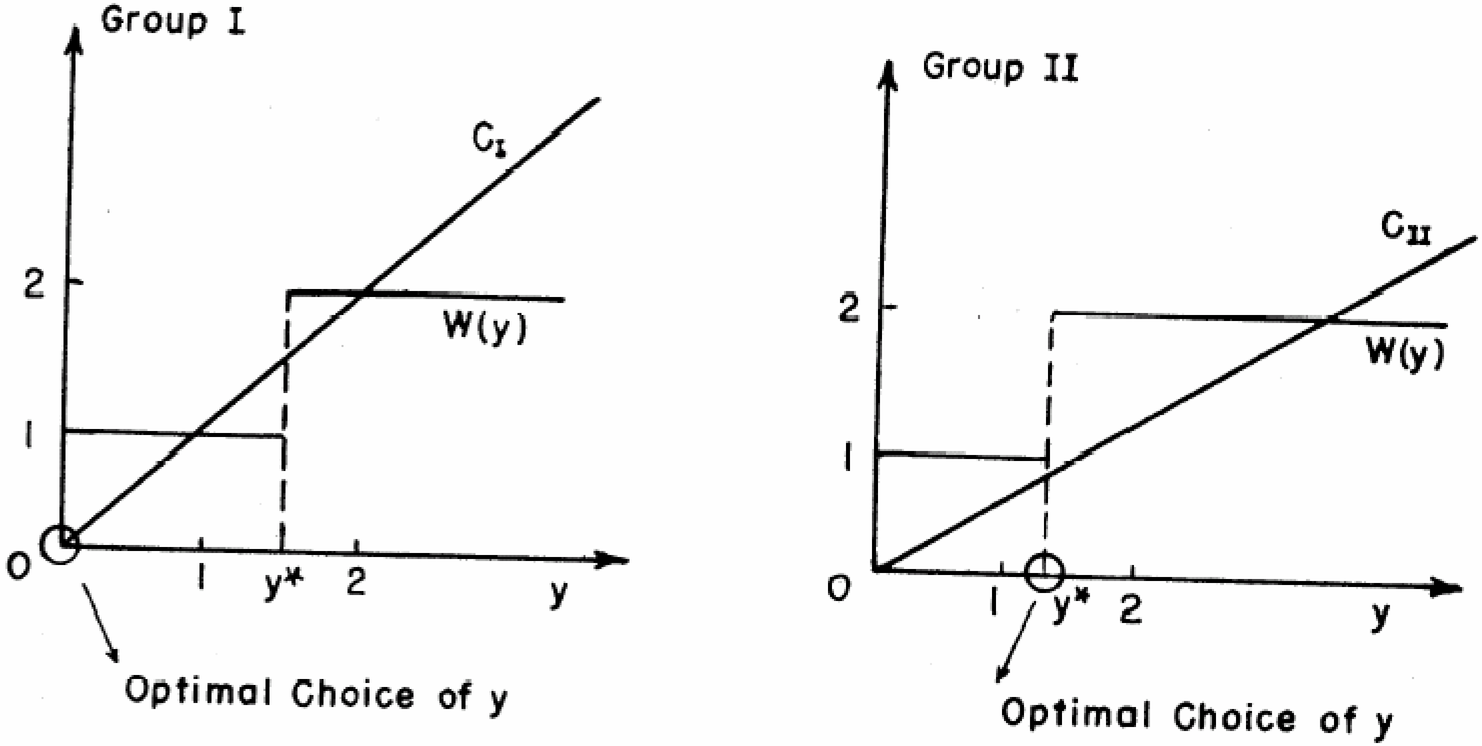

FIGURE III

Optimizing Choice of Education for Both Groups

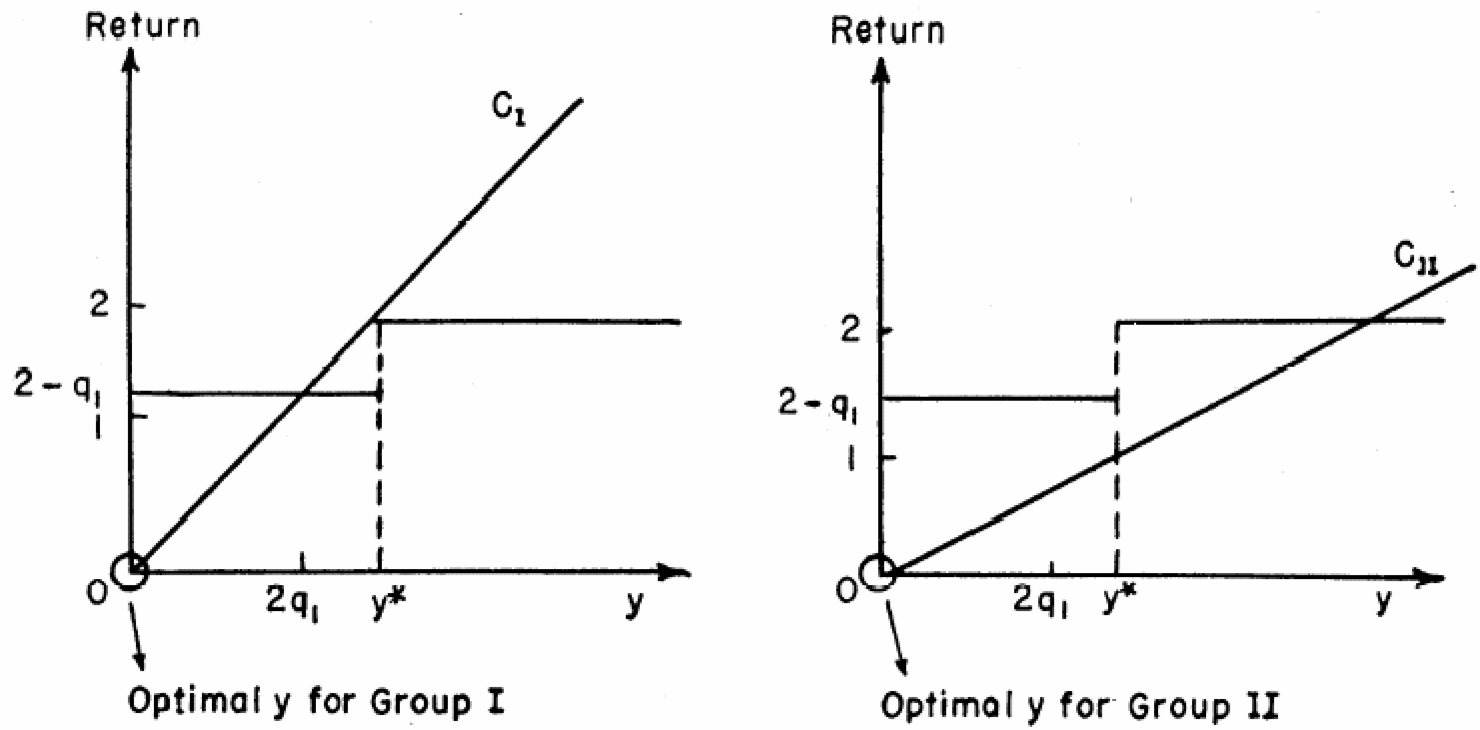

FIGURE IV

Optimal Signaling Decisions for the Two Groups

In addition, signaling games always have a “babbling equilibrium”

The speaker sends signals independent of his type and the receiver makes decisions independent of the signal.

When Talk is Cheap

- A cheap talk model is a signaling model where speech does not directly affect payoffs.
- In the seminal signaling model, a speech act was the purchase of a formal education that imposed direct costs on the speaker.
- In cheap talk models, a speech act does not require the purchase of any such good.

Crawford and Sobel (1982)

- There is a speaker and a receiver.
- The receiver makes a choice.
- Before the receiver chooses, a speaker advises the receiver about the consequences of her choice.
- Unlike the receiver, the speaker knows the consequences of the receiver's actions.
- Conclusion: “[E]quilibrium signaling is more informative when agents’ preferences are more similar.”

There are two players, a Sender (S) and a Receiver (R); only S has private information. S observes the value of a random varible, m, whose differentiable probability distribution function, $F(m)$, with density $f(m)$, is supported on $[0,1]$. S has a twice continuously differentiable von Neumann–Morgenstern utility function $U^S(y,m,b)$, where y, a real number, is the action taken by R upon receiving S's signal and b is a scalar parameter we shall later use to measure how nearly agents' interests coincide. R's twice continuously differentiable von Neumann–Morgenstern utility function is denoted $U^R(y,m)$. All aspects of the game except m are common knowledge.

Crawford and Sobel (1982)

- Here, all equilibria are partition equilibria and can be stated in terms that describe the accuracy of the speaker's statements.

- They (1441) conclude that "the more nearly [the speaker's and receiver's] interests coincide -- the finer partition there can be...As [the distance in their interests goes to infinity], [the number of partitions] eventually falls to unity and only the completely uninformative equilibrium remains."

- Corollary 1: this number goes to unity (the speaker's statement is totally uninformative) for even relatively small interest conflicts.

Intuition

- If S and R have common interests, then the speaker has an incentive to reveal what he knows and the receiver should believe what she hears.

- If what is good for a speaker is bad for a receiver, and vice versa, then high are the opportunity costs of speaking (as compared to saying nothing) or following a speaker's advice (as opposed to ignoring it).
  - In this case, the speaker has an incentive to reveal nothing and the receiver has an incentive to ignore everything.

Sequential Equilibrium

- A pair of beliefs and strategies is **consistent** $\Leftrightarrow$ the beliefs are the limit of a sequence of belief-strategy pairings such that the strategies are completely mixed and converge to the equilibrium strategy and the beliefs are calculated from the corresponding strategies by using Bayes' Theorem.

- A sequential equilibrium is a set of beliefs and strategies for all players that is both sequentially rational and consistent.
- Every extensive-form game has at least one sequential equilibrium.
- All PBE are SE, almost all SE are PBE. SE easier.

Morrow, Table 7.1

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Perfect Bayesian	At all information sets	Seq. Rationality at all Info sets	Yes	Can be chosen.
Sequential	At all information sets	Seq. Rationality at all Info sets	Yes	Consistent with trembles
Perfect	At all information sets	Against trembles. No Weak. Dom. S.	No	Irrelevant