Gibbons' Presentational Strategy

Information	complete		incomplete	
Sequence	static	dynamic	static	dynamic
Appropriate Nash Equilibrium concept	Generic	Subgame perfect	Bayesian or Bayes-Nash	Perfect Bayesian, sequential

- The equilibrium concepts build upon those of simpler games.
- Each subsequent concept, while more complex, also allows more precise conclusions from increasingly complex situations

Games of Incomplete Information

- In a game of incomplete information at least one player is uncertain about an aspect of another's utility function.
- i's utility is $u_i(a_1, \ldots a_n; t_i)$ where t_i is called player i's type and belongs to a set of possible types.
- Each type t_i corresponds to a different utility function that i might have.
- t_{-i} denotes others' types and $p(t_{-i}|t_i)$ denote i's belief about other types given her own type t_i.

Static Bayesian Games

Everyone is aware of the
informational uncertainties of others

Standard Assumptions

- It it is common knowledge that Nature draws a type vector $t=(t_1,\ldots t_n)$ according to the prior probability distribution p(t).
- Each player's type is the result of an independent draw.
- Players are capable of Bayesian updating.

Bayes' Theorem

- A: state of the world. *B*: an event.
- Conditional probability $p(B|A)$, is the likelihood of *B* given *A*.
- We use Bayes' Theorem to deduce the conditional probabilities of *A* given *B*.
- Bayes Theorem. If $(A_i)_{i=1,\ldots,n}$ is the set of states of the world and B is an event, then $p(A_i|B)=$

$$\frac{p(A_i)\,p(B\mid A_i)}{\sum_{i=1}^{n} p(A_i)\,p(B\mid A_i)}$$

- Know:
  - The prior belief is $p(A)$
  - The posterior belief is $p(A|B)$.

Strategy

- In the game $G=\{A_1,\ldots,A_n; t_1,\ldots t_n; p_1,\ldots,p_n; u_1,\ldots,u_n\}$, a strategy for i is a function $s_i(t_i)$, where for each type $t_i \in T_i$, $s_i(t_i)$ specifies the action from the feasible set A_i, that type t_i would choose if drawn by nature.

- Separating strategy: each type $t_i \in T_i$ chooses a different action $a_i \in A_i$.

- Pooling strategy, all types choose the same action.

- When deciding what to do, player i must think about what he or she would have done if each of the other types in T_i had been drawn.

Bayes-Nash Equilibrium

- In the static Bayesian game $G=\{A_1,\ldots,A_n;t_1,\ldots t_n; p_1,\ldots,p_n;u_1,\ldots,u_n\}$, the strategies $s^*=(s^*_1,\ldots,s^*_n)$ are a pure strategy Bayesian-Nash equilibrium if for each player i and for each of i's types $t_i \in T_i$, $s^*_i(t_i)$ solves $max_{\,ai \in Ai}\, \Sigma_{t\text{-}i \in T\text{-}I}\, u_i(s^*_1(t_1),\ldots,s_{i-1}(t_{i-1}),a_i,s^*_{i+1}(T_{i+1}),\ldots s^*_n(t_n);t)p_i(t_{-i}|t_i)$.

- That is, no player wants to change his or her strategy, even if the change involves only one action by one type.

Example 1

- Find all the pure strategy Bayesian Nash equilibria in the following static Bayesian game:

Game 1

	L	R
T	1,1	0.0
B	0,0	0,0

Game 2

	L	R
T	0,0	0,0
B	0,0	3,3

- Nature determines whether the payoffs are as in Game 1 or as in Game 2, each game being equally likely.
- Only Player 1 observes Nature's draw.
- Player A_1:{T,B}, A_2:{L,R}.

Example 1

- Candidates:

Row t_1	Row t_2	Col.	B-NE?
T	T	L	YES
T	T	R	NO
T	B	L	NO
T	B	R	YES
B	T	L	NO
B	T	R	NO
B	B	L	NO
B	B	R	YES

Example 2

- In the following normal form game, it is common knowledge that only the row player knows K with certainty and that Nature sets K=10 or K=-10 with equal probability. Find all of the Bayesian-Nash equilibria.

0,0	-K, 5
K/2, 10	5, 0

Example 2

0,0	-K, 5
K/2, 10	5, 0

- Row has a dominant strategy.
  - Bottom if “type” is K=10, Top if “type” is K=-10.
- Column’s best response is Left.
  - Any positive mass on “Right” reduces expected utility.
- What would happen if moves sequential, Row goes first?
  - Column observes Row’s move.
  - Column does not observe Row’s move.

Dynamic Bayesian Games

Everyone is aware of the informational uncertainties of others and thinks about implications for the past, present and future

Romer and Rosenthal (1978)

- Also see Niskanen (1971).
- How much does elite competition affect collective outcomes?
- Does monopoly proposal power have the same kind of effect in politics that it does in economics?

R&R Premises

- There are two players: an agenda setter and a median voter.
- There exists a status quo policy, $Q \in [0, 100]$.
- The setter makes a proposal $X \in [0, 100]$.
- The voter chooses a winner $Y \in \{X, Q\}$.
- Each player has an ideal point and single peaked preferences
  - $U_S = -|Y-S|$
  - $U_V = -|Y-V|$

R&R Conclusions

- Suppose $V \leq Q$ (parallel solution for $V > Q$.) In equilibrium, the voter will choose X only if $X \in [V-(|V-Q|), Q]$.
- The setter's best response to his anticipation of voter reactions is
  - If $S \in [V-(|V-Q|), Q]$, then $X=S=Y$.
  - If $S \in [0, V-(|V-Q|))$, then $X=\max[0, V-(|V-Q|)]=Y$.
  - If $S \in (Q, 1]$, then $X=S \neq Y$.
    - If we add a small cost of making a proposal, the setter makes no proposal in this case.

R&R Conclusion

S	V	Q	Range	Outcome
1	5	9		
3	7	1		
5	9	3		
7	1	5		
6	8	1		
1	2	3		
6	5	4		
8	7	9		

R&R Conclusion

S	V	Q	Range	Outcome
1	5	9	1-9	1
3	7	1	1-13	3
3	9	5	5-13	**5**
7	1	5	0-5	**5**
6	8	1	1-15	6
1	2	3	1-3	1
9	5	4	4-6	**6**
3	7	9	5-9	**5**

Example 3

- The voter is uncertain about the sender's type.
  - It is common knowledge that p(S=3)=.5=p(S=7).

V	Q	Range	X if S=3	X if S=7	Y
6	8				
7	7				
0	4				
3	0				

Example 3

- The voter is uncertain about the sender's type.
  - It is common knowledge that p(S=3)=.5=p(S=7).

V	Q	Range	X if S=3	X if S=7	Y
6	8	4-8	3	7	X,Q
7	7	7	3	7	Q
0	4	0-4	3	7	Q
3	0	0-6	3	7	X

Example 4

- It is common knowledge that p(S=3)=.5=p(S=7)
  - The setter must pay a cost of 2 to make a proposal. If he makes no proposal, SQ is the outcome.

V	Q	Range	X if S=3	X if S=7	Out come
6	8				
7	7				
0	4				
3	0				

Example 4

- It is common knowledge that p(S=3)=.5=p(S=7)
  - The setter must pay a cost of 2 to make a proposal. If he makes no proposal, SQ is the outcome.

V	Q	Range	X if S=3	X if S=7	Out come
6	8	4-8	3	NO	**Q**
7	7	7	NO	NO	Q
0	4	0-4	3	NO	**X**
3	0	0-6	3	7	X

Do the game on Gibbons 176.

Look for Nash Equilibria, then
discuss the problem with the answer.

Perfect Concepts

- The subgame perfect equilibrium concept adds to the Nash concept the requirement that players choose optimally in subgames.
- But, a proper subgame cannot begin at an information set containing multiple nodes.
- Perfect Bayesian equilibrium adds to Nash the requirement that players choose optimally given their beliefs about the rest of the game.

Sequential Rationality

- A pair of beliefs and strategies is **sequentially rational** iff from each information set, the moving player's strategy maximizes its expected utility for the remainder of the game given its beliefs and all players' strategies.
- Sequential rationality allows a process akin to backwards induction on games of incomplete information.

Perfect Bayesian Equilibrium

- A perfect Bayesian equilibrium is a belief-strategy pairing such that
  - the strategies are sequentially rational given the beliefs
  - and the beliefs are calculated from the equilibrium strategies by Bayes' Theorem whenever possible.
- A defection from the equilibrium path does not increase the chance that others will play "irrationally."
- Every finite n-person game has at least one perfect Bayesian equilibrium in mixed strategies.

Implications

- In a PBE, players cannot threaten to play strategies that are strictly dominated beginning at any information set off the equilibrium path.

- A single pass working backwards through the tree (typically) will not suffice to compute a PBE.

Do the game on Gibbons 181.

Look for Nash Equilibria, then discuss the problem with the answer.

Do Exercise 6.10 in Morrow

Requirements for PBE in Extensive-Form Games

- An information set is **on the equilibrium path** if it will be reached with positive probability ⇔ the game is played according to the equilibrium strategies.

- On the equilibrium path, Bayes' Rule and equilibrium strategies determine beliefs.

- *Off* the equilibrium path, Bayes' Rule and equilibrium strategies determine beliefs *where possible*.