1. Consider the function

\[ p(x) = \begin{cases} 
0, & \text{if } x < 0 \\
kx^2(1-x), & \text{if } 0 \leq x \leq 1 \\
0, & \text{if } x > 1 
\end{cases} \]

(a) For what value of \( k \) is \( p(x) \) a probability density function? (3 pts)

We have

\[
1 = \int_{-\infty}^{\infty} p(x)dx = \int_{0}^{1} kx^2(1-x)dx = \int_{0}^{1} k(x^2 - x^3)dx = k \left( \frac{x^3}{3} - \frac{x^4}{4} \right) \bigg|_0^1 = k \frac{1}{12}.
\]

Thus \( k = 12 \).

(b) Using that value of \( k \), find the probability that \( x \) is greater than 0.5. (2 pts)

We set up

\[
\int_{0.5}^{\infty} p(x)dx = \int_{0.5}^{1} 12(x^2 - x^3)dx = 12 \left( \frac{x^3}{3} - \frac{x^4}{4} \right) \bigg|_{0.5}^{1} = \frac{11}{16}.
\]

(c) Find the mean. (3 pts)

We set up and calculate

\[
\text{mean of } x = \int_{-\infty}^{\infty} xp(x)dx = \int_{0}^{1} 12x(x^2 - x^3)dx = \int_{0}^{1} 12(x^3 - x^4)dx = 12 \left( \frac{x^4}{4} - \frac{x^5}{5} \right) \bigg|_0^1 = \frac{12}{20} = 0.6.
\]

(d) Find the median. (Hint: Use your calculator, but not do do any definite integrals.) (3 pts)

We set up the definition of the median \( T \) and obtain

\[
0.5 = \int_{-\infty}^{T} p(x)dx = \int_{0}^{T} 12(x^2 - x^3)dx = 12 \left( \frac{T^3}{3} - \frac{T^4}{4} \right).
\]

Using a calculator to solve numerically using either the root or intersect function yields \( T \approx 0.61427 \).
2. Use the formulas for the sums of geometric series on the following. (3 pts each)

(a) \(1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots\)

The infinite geometric series formula is \(\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}\). This series has \(a = 1\) and \(x = -\frac{1}{2}\) and so the sum is
\[
\frac{1}{1 - (-\frac{1}{2})} = \frac{1}{\left(\frac{3}{2}\right)} = \frac{2}{3}
\]

(b) \(\sum_{n=5}^{15} \left(\frac{2}{3}\right)^n\) The finite geometric series formula is \(\sum_{i=0}^{n-1} ax^i = \frac{a(1-x^n)}{1-x}\).
\[
\left(\frac{2}{3}\right)^5 + \left(\frac{2}{3}\right)^6 + \cdots + \left(\frac{2}{3}\right)^{15} = \left(\frac{2}{3}\right)^5 \left(1 + \left(\frac{2}{3}\right) + \cdots + \left(\frac{2}{3}\right)^{10}\right)
\]
\[
= \left(\frac{2}{3}\right)^5 \left(\frac{1 - \left(\frac{2}{3}\right)^{11}}{1 - \left(\frac{2}{3}\right)}\right); \]
where we used the finite geometric series formula with \(a = 1\), \(x = \frac{2}{3}\), and \(n-1 = 10\) once we factored out the \(\left(\frac{2}{3}\right)^5\).

3. Suppose that you make monthly deposits into a savings account of $250, with the first deposit occurring today. Every month, your account pays 4% interest. Let \(B_n\) represent the balance in your account immediately after the \(n\)th deposit.

(a) Find \(B_1\), \(B_2\), and \(B_3\). (3 pts) With each deposit, the balance increases by $250. Moreover, the amount in the account at the time of the previous deposit has increased by 4%, therefore is multiplied by 1.04. We have
\[
B_1 = 250, \quad B_2 = 250 + B_1 \cdot 1.04 = 250 + 250(1.04)
\]
\[
B_3 = 250 + B_2 \cdot 1.04 = 250 + (250+250(1.04))1.04 = 250+250(1.04)+250(1.04)^2
\]

(b) Find a closed form, explicit formula (that is, no summation signs or + \cdots +) for \(B_n\). (5 pts) We extrapolate the pattern and observe
\[
B_n = 250 + 250(1.04) + 250(1.04)^2 + \cdots + 250(1.04)^{n-1}.
\]
Using the finite geometric series formula with \(a = 250\) and \(x = 1.04\) we obtain
\[
B_n = 250 \frac{\left(1 - 1.04^n\right)}{\left(1 - 1.04\right)}.
\]