1 Problem 1

Consider a restricted three body problem defined as follows.

1. Assume a satellite always remains at a fixed point on the line between the Earth and the Moon so that the gravitational force of the Earth on the satellite equals the gravitational force of the Moon on the satellite. What is the fixed distance of the satellite from the Earth? What is the fixed distance of the satellite from the Moon? Show that the orbit of the satellite about the Earth is circular. What is the period of the satellite.

2. Assume a satellite always remains at a fixed point on the line between the Sun and the Earth so that the gravitational force of the Sun on the satellite equals the gravitational force of the Earth on the satellite. What is the fixed distance of the satellite from the Sun? What is the fixed distance of the satellite from the Earth? Show that the orbit of the satellite about the Sun is circular. What is the period of the satellite?

1.1 Earth-Moon balance

The gravitational force balance on a satellite of mass \( m \) is

\[
\frac{m \mu_e}{r_e^2} = \frac{m \mu_m}{r_m^2} \tag{1}
\]

where \( r_e \) is the distance from the Earth’s center, \( r_m \) is the distance from the Moon’s center, \( \mu_e \) is the Earth’s gravitational parameter and \( \mu_m \) is the Moon’s gravitational parameter. Since the Earth-Moon distance \( r_e/m = r_e + r_m \) is fixed, Eqn. 1 becomes

\[
\mu_m r_e^2 = \mu_e (r_e/m - r_e)^2
\]

This can be rearranged as a quadratic equation

\[
(\mu_m - \mu_e) r_e^2 + 2 \mu_e r_e/m r_e - \mu_e r_e^2/m = 0
\]

which has roots at

\[
r_e = \frac{\mu_e r_e/m}{\mu_e - \mu_m} \left( 1 \pm \sqrt{\frac{\mu_m}{\mu_e}} \right)
\]
Since $\mu_e/\mu_m = M_e/M_m = 81.30$,

$$\frac{\mu_e}{\mu_e - \mu_m} = \frac{1}{1 - 1/81.30} = 1.0125$$

so that

$$\frac{r_e}{r_{e/m}} = 1.0125(1 \pm \sqrt{1/81.30}) = 1.0125 \pm 0.11229$$

is only less than one (and thus physical) for the smaller root. For $r_{e/m} = 384,400$ km$^1$,

$$r_e = (1.0125 - 0.11229)384,400 \text{ km} = 346,024 \text{ km}$$

and

$$r_m = r_{e/m} - r_e = (384,400 - 346,024) \text{ km} = 38,376 \text{ km}$$

Since $r_e$ is constant, the satellite has a circular orbit around the Earth with the same period as the Moon, 27.317 days.

### 1.2 Earth-Sun balance

As before, the gravitational force balance on a satellite of mass $m$ is

$$\frac{m\mu_e}{r_e^2} = \frac{m\mu_s}{r_s^2}$$

where $r_e$ is the distance from the Earth’s center, $r_s$ is the distance from the Sun’s center, $\mu_e$ is the Earth’s gravitational parameter and $\mu_m$ is the Sun’s gravitational parameter. Since the Sun-Moon distance $r_{e/s} = r_e + r_s$ is fixed, Eqn. 2 becomes

$$\mu_s r_e^2 = \mu_e (r_{e/s} - r_e)^2$$

This can be rearranged as a quadratic equation

$$(\mu_s - \mu_e)r_e^2 + 2\mu_e r_{e/s}r_e - \mu_e r_{e/s}^2 = 0$$

which, as before, has only one physical root at

$$r_e = \frac{\mu_e r_{e/s}}{\mu_e - \mu_s} \left(1 - \sqrt{\frac{\mu_s}{\mu_e}}\right) = \frac{3.9860 \times 10^5(1.4960 \times 10^8)}{3.9860 \times 10^8 - 1.3272 \times 10^{11}} \left(1 - \sqrt{\frac{1.3272 \times 10^{11}}{3.9860 \times 10^8}}\right) \text{ km} = 258,809 \text{ km}$$

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and
\[ r_s = r_{e/s} - r_e = (1.4960 \times 10^8 - 258,809) \text{ km} = 1.4934 \times 10^8 \text{ km} \]

Since \( r_s \) is constant, the satellite has a circular orbit around the Sun with the same period as the Earth, 364.25 days.

2 Problem 2

An Earth artificial satellite in elliptical orbit is observed to have an altitude at perigee of 220 km and an altitude at apogee of 360 km.

1. What is the orbital specific angular momentum? What is the orbital specific total energy?
2. What is the orbital eccentricity?
3. What are the specific kinetic energy and specific potential energy of the orbit at perigee?
4. What are the specific kinetic energy and specific potential energy of the orbit at apogee?
5. What is the orbital velocity at perigee? What is the orbital velocity at apogee?
6. At a true anomaly of 45 degrees as measured from perigee, determine the orbital altitude, the orbital velocity, and the flight path angle.
7. What is the orbital period?

2.1 Specific angular momentum & total energy

Since the Earth’s mean equatorial radius \( r_0 = 6378.1 \text{ km} \), the semi-major axis

\[ a = \frac{1}{2}(r_a + r_p) = \frac{1}{2}(6598.1 + 6738.1) \text{ km} = 6668.1 \text{ km} \]

The specific total energy is then

\[ E = \frac{-\mu}{2a} = \frac{3.9860 \times 10^5 \text{ km}^2}{2(6668.1) \text{ s}^2} = -29.889 \text{ km}^2 \text{ s}^{-2} \]

Since the eccentricity

\[ e = \frac{r_a - r_p}{r_a + r_p} = \frac{6738.1 - 6598.1}{6738.1 + 6598.1} = 1.0498 \times 10^{-2} \quad (3) \]

the semi-latus rectum

\[ p = a(1 - e^2) = 6668.1(0.999989) \text{ km} = 6667.4 \text{ km} \]

so the specific angular momentum

\[ H = \sqrt{\mu p} = \sqrt{6677.4(3.9860 \times 10^5)} \frac{\text{ km}^2}{\text{ s}} = 51,552 \frac{\text{ km}^2}{\text{ s}} \]
2.2 Orbital eccentricity

As shown in Eqn. 3 above, the eccentricity \( e = 1.0498 \times 10^{-2} \).

2.3 Specific kinetic & potential energy at perigee

At perigee, the velocity

\[
v_p = \frac{H}{r_p} = \frac{51,552}{6598.1} \frac{\text{km}}{\text{s}} = 7.8132 \frac{\text{km}}{\text{s}}
\]  

so the specific kinetic energy

\[
K_p = \frac{1}{2} v_p^2 = \frac{(7.8132)^2}{2} \frac{\text{km}^2}{\text{s}^2} = 30.522 \frac{\text{km}^2}{\text{s}^2}
\]

while the specific potential energy

\[
V_p = -\frac{\mu}{r_p} = -\frac{3.9860 \times 10^5}{6598.1} \frac{\text{km}^2}{\text{s}^2} = -60.411 \frac{\text{km}^2}{\text{s}^2}
\]

Adding these components gives the specific total energy

\[
\mathcal{E} = K_p + V_p = (30.522 - 60.411) \frac{\text{km}^2}{\text{s}^2} = -29.888 \frac{\text{km}^2}{\text{s}^2}
\]

which agrees with the value given in section 2.1.

2.4 Specific kinetic & potential energy at apogee

At apogee, the velocity

\[
v_a = \frac{H}{r_a} = \frac{51,552}{6738.1} \frac{\text{km}}{\text{s}} = 7.6508 \frac{\text{km}}{\text{s}}
\]  

so the specific kinetic energy

\[
K_a = \frac{1}{2} v_a^2 = \frac{(7.6508)^2}{2} \frac{\text{km}^2}{\text{s}^2} = 29.268 \frac{\text{km}^2}{\text{s}^2}
\]

while the specific potential energy

\[
V_a = -\frac{\mu}{r_a} = -\frac{3.9860 \times 10^5}{6738.1} \frac{\text{km}^2}{\text{s}^2} = -58.156 \frac{\text{km}^2}{\text{s}^2}
\]
Adding these components gives the specific total energy

\[
\mathcal{E} = K_a + V_a = (29.268 - 58.156) \frac{\text{km}^2}{\text{s}^2} = -29.888 \frac{\text{km}^2}{\text{s}^2}
\]

which still agrees with the value given in section 2.1.

### 2.5 Orbital velocity at perigee & apogee

As shown in Eqn. 4 and 5, \( v_p = 7.8132 \text{ km/s} \) and \( v_a = 7.6508 \text{ km/s} \).

### 2.6 Orbital parameters at \( \theta = 45^\circ \)

At true anomaly \( \theta = 45^\circ \), the altitude

\[
h_{45} = \frac{p}{1 + e \cos \theta} - r_\oplus = \left( \frac{6667.4}{1 + 1.0498 \times 10^{-2} \cos 45^\circ} - 6378.1 \right) \text{ km} = 240.17 \text{ km}
\]

Rearranging the \( \mathcal{E} \)-equation, the velocity

\[
v_{45} = \sqrt{\frac{2 (\mathcal{E} + \frac{\mu}{r_{45}})}{\mu}} = \sqrt{2 \left( \frac{6667.4}{6618.3} - 29.844 + \frac{3.8960 \times 10^5}{6618.3} \right)} \text{ km/s} = 7.7953 \text{ km/s}
\]

The tangential component of this is

\[
r_\theta = H \frac{51.552}{6618.3} \text{ km/s} = 7.7893 \text{ km/s}
\]

so the flight path angle

\[
\gamma = \cos^{-1} \left( \frac{r_\theta}{v_{45}} \right) = \cos^{-1} \left( \frac{7.7919}{7.7921} \right) = 2.246^\circ
\]

### 2.7 Orbital period

The orbital period for this satellite is

\[
T = 2\pi \sqrt{\frac{a^3}{\mu}} = 2\pi \sqrt{\frac{(6668.1)^3}{3.9860 \times 10^5}} \text{ s} = 5418.9 \text{ s} = 1 \text{ hr 30 min 18.9 s}
\]
3 Problem 3

It is known that Halley’s comet has a period of 76 years and its closest approach to the Sun is 90 million km.

1. What is the orbital specific angular momentum of Halley’s comet? What is the orbital specific total energy of Halley’s comet?
2. What is the orbital eccentricity of Halley’s comet?
3. What are the aphelion and perihelion of the orbit of Halley’s comet?
4. What is the orbital velocity of Halley’s comet at perihelion? What is the orbital velocity of Halley’s comet at aphelion?

3.1 Specific angular momentum & total energy

Rearranging Eqn. 6, the semimajor axis

\[
a = \left[ \frac{\mu}{\left( \frac{T}{2\pi} \right)^2} \right]^{1/3} = \left[ 1.3272 \times 10^{11} \left( \frac{2.3984 \times 10^9}{2\pi} \right)^2 \right]^{1/3} \text{ km} = 2.6841 \times 10^9 \text{ km}
\]

so the specific total energy

\[
\mathcal{E} = -\frac{\mu}{2a} = -\frac{1.3272 \times 10^{11}}{2(2.6841 \times 10^9)} \frac{\text{km}^2}{\text{s}^2} = -24.723 \frac{\text{km}^2}{\text{s}^2}
\]

Given perihelion at \(r_p = (6.9600 \times 10^5 + 9.0000 \times 10^7) \text{ km} = 9.0696 \times 10^7 \text{ km} \), the eccentricity

\[
e = 1 - \frac{r_p}{a} = 1 - \frac{9.0696 \times 10^7}{2.6841 \times 10^9} = 0.96621
\]

The semi-latus rectum

\[
p = a(1 - e^2) = 2.6841 \times 10^9(1 - [0.96621]^2) \text{ km} = 1.7833 \times 10^8 \text{ km}
\]

so the specific angular momentum

\[
H = \sqrt{\mu \rho} = \sqrt{1.3272 \times 10^{11}(1.7833 \times 10^8)} \frac{\text{km}^2}{\text{s}} = 4.8649 \times 10^9 \frac{\text{km}^2}{\text{s}}
\]

3.2 Orbital eccentricity

As shown above in Eqn. 7, the eccentricity \(e = 0.96621\).
### 3.3 ApHELION & PERIHELION

Given perihelion \( r_p = 9.0696 \times 10^7 \text{ km} \), the aphelion

\[
r_a = a(1 + e) = 2.6841 \times 10^9(1.96621) \text{ km} = 5.2775 \times 10^9 \text{ km}
\]

### 3.4 Orbital velocity at perihelion & aphelion

At perihelion, the orbital velocity

\[
v_p = \frac{H}{r_p} = \frac{4.8649 \times 10^9 \text{ km}}{9.0696 \times 10^7 \text{ s}} = 53.64 \frac{\text{km}}{\text{s}}
\]

while the aphelion orbital velocity

\[
v_a = \frac{H}{r_a} = \frac{4.8649 \times 10^9 \text{ km}}{5.2775 \times 10^9 \text{ s}} = 0.92182 \frac{\text{km}}{\text{s}}
\]

### 4 Problem 4

At the end of a rocket launch of a space vehicle from a launch site on the Earth’s equator, the burnout velocity is 12 km/sec in direction due east in the plane of the equator; at burnout the flight path angle is 2 degrees above the horizontal and the burnout altitude is 120 km.

1. What is the orbital specific angular momentum? What is the orbital specific total energy?
2. What is the orbital eccentricity?
3. What is the equation of the conic section that describes the orbit? Assume the angle \( \theta \) is measured relative to the radius vector at burnout, with increasing \( \theta \) taken in the direction of the motion of the space vehicle.
4. What are the perigee and apogee of the orbit?
5. What is the orbital velocity at perigee? What is the orbital velocity at apogee?
6. What is the orbital period?

### 4.1 Specific angular momentum & total energy

The orbital specific angular momentum

\[
H = |\hat{r}_b \times \hat{v}_b| = r_b v_b \cos \gamma = (6378.1 + 120)12 \cos 12^\circ \frac{\text{km}^2}{\text{s}} = 77,930 \frac{\text{km}^2}{\text{s}}
\]

while the orbital specific total energy

\[
E = \frac{1}{2} v_b^2 - \frac{\mu}{r_b} = \frac{(12)^2}{2} - \frac{3.9860 \times 10^5 \text{ km}^2}{6498.1 \text{ s}^2} = 10.659 \frac{\text{km}^2}{\text{s}^2}
\]
4.2 Orbital eccentricity

The orbital eccentricity

\[ e = \sqrt{1 + \frac{2\varepsilon H^2}{\mu_{\oplus}^2}} = \sqrt{1 + \frac{2(10.659)(77,930)^2}{(3.9860 \times 10^5)^2}} = 1.3472 \]

4.3 Equation of orbit

The semi-latus rectum

\[ p = \frac{H^2}{\mu_{\oplus}} = \frac{(77,930)^2}{3.9860 \times 10^5} \text{ km} = 15,236 \text{ km} \]

so the conic equation is

\[ r = \frac{p}{1 + e \cos(\theta - \theta_0)} = \frac{15,236}{1 + 1.3472 \cos(\theta - \theta_0)} \text{ km} \]

Substituting the burnout radius into the above equation,

\[ r_b = \frac{p}{1 + e \cos(-\theta_0)} \]

which can be rearranged to give the true anomaly at burnout

\[ \theta_0 = \cos^{-1} \left( \frac{p/r_b - 1}{e} \right) = \cos^{-1} \left( \frac{15,236/6498.1 - 1}{1.3472} \right) = 3.5013^\circ \]

and, thus, the full conic equation

\[ r = \frac{p}{1 + e \cos(\theta - \theta_0)} = \frac{15,236}{1 + 1.3472 \cos(\theta - 3.5013^\circ)} \text{ km} \]

4.4 Perigee & apogee

Perigee is at

\[ r_p = \frac{15,236}{1 + 1.3472} \text{ km} = 6491.2 \text{ km} \]

or an altitude of \( h_p = r_p - r_{\oplus} = (6491.2 - 6378.1) \text{ km} = 113.1 \text{ km} \). Since \( e = 1.3472 > 1 \), this is a hyperbolic orbit and the apogee

\[ r_a \rightarrow \infty \]
4.5 Orbital velocity at perigee & apogee

At perigee, the orbital velocity

\[ v_p = \frac{H}{r_p} = \frac{77,930 \text{ km}}{6498.1 \text{ s}} = 11.993 \text{ km/s} \]

while the apogee orbital velocity

\[ v_a = \sqrt{2 \left( E + \frac{\mu}{r_a} \right)} = \sqrt{2(10.659)} \text{ km/s} = 4.6171 \text{ km/s} \]

4.6 Orbital period

Since this is an open orbit, the orbital period

\[ T \to \infty \]

5 Problem 5

A space object enters the sensible atmosphere of the Earth at an altitude of 350,000 ft with a velocity of 23,000 ft/sec and a flight path angle of -63 degrees.

1. What type of orbit is this?

2. What were the velocity and flight path angle of the space object when it was at an altitude of 100 mi during descent?

3. Ignoring the effect of the Earth’s atmosphere on the motion of the object, determine if the object impacts the surface of the Earth or not. If it does, what is its impact velocity? If not, what is its minimal miss distance.

5.1 Type of orbit

As before, start with the specific total energy at \( r = r_E + h = (2.0926 \times 10^7 + 3.5000 \times 10^5) \text{ ft} = 2.1276 \times 10^7 \text{ ft}, \)

\[ E = \frac{1}{2} v^2 - \frac{\mu_{\text{Earth}}}{r} = \left[ \frac{(23,000)^2}{2} - \frac{1.4076 \times 10^{16}}{2.1276 \times 10^7} \right] \frac{\text{ft}^2}{\text{s}^2} = -3.9709 \times 10^8 \frac{\text{ft}^2}{\text{s}^2} \]

Since \( E < 0 \), the orbit is elliptical.

5.2 Velocity at 100 mi altitude

At 100 mi altitude, \( r = (3963.2 + 100)5280 \text{ ft} = 2.1454 \times 10^7 \text{ ft} \), so the velocity magnitude
\[ v = \sqrt{2 \left( \mathcal{E} + \frac{\mu_B}{r} \right)} = \sqrt{2 \left( -3.9709 \times 10^8 + \frac{1.4076 \times 10^{16}}{2.1454 \times 10^7} \right)} \text{ ft/s} = 22.761 \text{ ft/s} \]

5.3 Hit or miss?

The specific angular momentum can be found from \( \vec{r} \times \dot{\vec{v}} \) at 100 mi altitude,

\[ H = rv \cos \gamma = [2.1276 \times 10^7 (23,000) \cos(-63^\circ)] \frac{\text{ft}^2}{\text{s}} = 2.2216 \times 10^{11} \frac{\text{ft}^2}{\text{s}} \]

The semi-latus rectum

\[ p = \frac{H}{\mu_B} = \frac{2.2216 \times 10^{11}}{1.4076 \times 10^{16}} \text{ ft} = 3.5063 \times 10^6 \text{ ft} \]

while the eccentricity

\[ e = \sqrt{1 + \frac{2\mathcal{E}h^2}{\mu_B^2}} = \sqrt{1 + \frac{2(-3.9709 \times 10^8)(2.2216 \times 10^{11})^2}{(1.4076 \times 10^{16})^2}} = 0.89564 \]

so the perigee

\[ r_p = \frac{p}{1+e} = \frac{3.5063 \times 10^6}{1.89564} \text{ ft} = 1.8497 \times 10^6 \text{ ft} \]

is less than the radius of the Earth, \( r_p < r_B \). Clearly, this means that the object hits at a speed of

\[ v = \sqrt{2 \left( \mathcal{E} + \frac{\mu_B}{r_B} \right)} = \sqrt{2 \left( -3.9709 \times 10^8 + \frac{1.4076 \times 10^{16}}{2.0926 \times 10^7} \right)} \text{ ft/s} = 23.476 \text{ ft/s} \]

with a tangential component

\[ r\dot{\theta} = \frac{H}{r_B} = \frac{2.2216 \times 10^{11}}{2.0926 \times 10^7} \frac{\text{ft}}{\text{s}} = 10,616 \frac{\text{ft}}{\text{s}} \]

giving a flight path angle at impact

\[ \gamma = -\cos^{-1} \left( \frac{r\dot{\theta}}{v} \right) = -\cos^{-1} \left( \frac{10,616 \dot{\theta}}{23,476} \right) = -63.113^\circ \]