Solutions to Homework Assignment 2

1. (Chapter 2, Exercise 35, Part b)
   \[ bb, ab, abb \]

2. Chapter 2, Exercise 36 b) and d)
   
   b) (There are many possibilities. Here are five). a, b, aab, bab, aabab.
   
   d) Note that the only string in \( \{a\}^* \cap \{b\}^* \) is the empty string. So we are really asking for strings in \( \{ab\}^* \). Here are five: \( \epsilon, ab, abab, ababab, abababab \).

3. (Chapter 2, Exercise 39.)
   
   Let \( X = \{\epsilon, aa\}, Y = \{aa\}, Z = \{aaaa\}. \)
   
   Then \( X \circ (Y \cap Z) = \{\epsilon, aa\} \circ \emptyset = \emptyset. \)
   
   But \( (X \circ Y) \cap (X \circ Z) = \{aa, aaaa\} \cap \{aaaa, aaaaaa\}. \)
   
   \[ = \{aaaa\}. \]

4. Chapter 2 Exercise 49 Part c) and e)
   
   c) Here are the first six values of the function with \( f(0) = 1 \) and \( f(n + 1) = f(n) \):
   
   \[ 1, 1, 1, 1, 1, 1 \]
   
   e) Here are the first six values of the function with \( f(0) = 1 \) and \( f(n + 1) = 2 + (n \cdot (f(n) - 1)) \):
   
   \[ 2, 3, 4, 8, 23, 90 \]

5. Answer: Base clause: \( f(0) = a^0 = 1 \); Recursion clause: \( f(n + 1) = a^{(n+1)} = a^n \cdot a \)

6. Chapter 2 Exercise 52

   **Base case**
   
   The only set with no members is \( \emptyset, \mathcal{P}(\emptyset) = \{\emptyset\} \) has 1 member. This is what we need to prove, since \( 2^0 = 1 \).

   **Induction Step**
   
   Assume that the thesis is true for every set \( X \) with \( n \) or fewer members. Say that \( X' \)
has $n + 1$ members. Then $X'$ isn’t empty, so we can pick some element $a \in X'$. Let $X'' = X' - \{a\}$. We know that $X''$ has $n$ members, so the induction hypothesis applies to $X''$: $P(X'')$ has $2^n$ members. Every subset of $X''$ is a subset of $X'$, since $X'' \subseteq X'$, and so $P(X'') \subseteq P(X')$.

Note that we can pair up all the elements of $P(X'')$ with elements of $P(X') - P(X'')$ so that every member of $P(X'')$ corresponds to exactly one member of $P(X') - P(X'')$: for any $Y \in P(X'')$, let its partner be $\{a\} \cup Y$, which will be a member of $P(X') - P(X'')$. This shows that there are the same number of elements in $P(X'')$ and in $P(X') - P(X'')$. But we already know that $P(X'')$ has $2^n$ elements, so $P(X') - P(X'')$ has $2^n$ elements as well.

Finally, we note that $P(X') = P(X'') \cup (P(X') - P(X''))$. Also, $P(X'')$ and $P(X') - P(X'')$ have no elements in common. (This point is crucial.) So the size of $P(X') = (\text{size of } P(X'')) + (\text{size of } (P(X') - P(X''))) = 2^n + 2^n = 2 \cdot 2^n = 2^{n+1}$.

This proves the induction step.

The base case and the induction step together prove the claim.

7. Chapter 2 Exercise 58

Inductive Definition of $X^*$: Base Clause: $\epsilon \in X^*$
Recursion Clause: If $x \in X^*$ and $y \in X$ then $xy \in X^*$
Closure Clause: Nothing else is in $X^*$ (It’s ok if you don’t make the closure clause explicit)

Proof that this defines the Kleene $^*$: Let $X^{\text{Inductive}*}$ be the set we’ve defined, and let $X^*$ be the Kleene $^*$ of $X$ according to the textbook definition. (i.e. $X^* = \{x_1 \ldots x_n / n \geq 0$ and $x_1, \ldots x_n \in X\}$. We want to prove that $X^{\text{Inductive}*} = X^*$. As usual, we prove containment both ways.

$\Rightarrow$ Say that $s \in X^{\text{Inductive}*}$. We want to show that $s \in X^*$. We can proceed by induction (on the number of symbols in $s$).
First note (base case) that if $s = \epsilon$, then $s \in X^*$ by definition.
Assume (induction hypothesis) that $s$ is not the empty string, and that for every string with fewer letters than $s$, if $s \in X^{\text{Inductive}*}$ then $s \in X^*$. From the inductive definition, we can see that there must be some $s' \in X^{\text{Inductive}*}$ and $\tilde{s} \in X$ such that $s = s' \tilde{s}$. Since $s'$ is shorter than $s$, we know that $s' \in X^*$ by the induction hypothesis. Hence by the definition of $X^*$ there are strings $x_1 \in X, x_2 \in X, \ldots, x_i \in X$ such that $s' = x_1 x_2 \ldots x_i$. But then $s = x_1 x_2 \ldots x_i \tilde{s}$. That is, $s$ consists of a concatenation of strings from $X$, so by definition $s$ is in $X^*$. This completes the inductive step.

The base step and the induction step allow us to conclude $s \in X^*$. Since $s$ was chosen arbitrarily, this proves that $X^{\text{Inductive}*} \subseteq X^*$. 

2
Say that \( s \in X^* \). We want to show that \( s \in X^{Inductive*} \). Again we proceed by induction (on the number of symbols in \( s \)).

First note (base case) that if \( s = \epsilon \), then \( s \in X^{Inductive*} \) by definition.

Assume (induction hypothesis) that \( s \) is not the empty string, and that for every string with fewer letters than \( s \), if \( s \in X^* \) then \( s \in X^{Inductive*} \). By the definition of \( X^* \), there are strings \( x_1 \in X, x_2 \in X, \ldots, x_{l-1} \in X, x_l \in X \) such that \( s = x_1x_2\ldots x_{l-1}x_l \). By the definition of \( X^* \), \( s' = x_1x_2\ldots x_{l-1} \in X^* \), so by the inductive hypothesis, \( s' \in X^{Inductive*} \). By the inductive definition of \( X^{Inductive*} \), \( s'x_l \in X^{Inductive*} \). Since \( s = s'x_l \), we have that \( s \in X^{Inductive*} \). This completes the inductive step.

The base step and the induction step allow us to conclude \( s \in X^{Inductive*} \). Since \( s \) was chosen arbitrarily, this proves that \( X^* \subseteq X^{Inductive*} \).

Combining \( \Leftarrow \) and \( \Rightarrow \), along with the second version of the principle of extensionality, gives: \( X^* = X^{Inductive*} \).