Elasticity and Stiffness

Ref. Theory of elasticity by Landau and Lifshitz.

Q1: Soft or hard, how should we describe it?
Q2: Why do we use cork oak to make bottle stoppers?
Q3: Why the sound velocity of the LA mode is always larger than the TA mode?

### 4.1. Hooke’s law

For a spring

\[ F = C \Delta L \]  

(4.1)

The elastic energy:

\[ E = \frac{1}{2} C \Delta L^2 \]  

(4.2)

Q: How to generalize this Hooke’s law to a 3D solid?

### 4.2. Strain tensor

#### 4.2.1. I.h.s. of the Hooke’s law

Let’s take a closer look at the r.h.s. of Eq. (4.1). \( \Delta L \) is the change of length of the spring. In other words, it is the change distance between point A and B, where A and B are the two ending points of the spring.

Q: For a 3D solid, how should we describe the change of distance between to points A and B?

A: the strain tensor

For a solid, we label every points in a crystal using its coordinates \( \mathbf{r} = (x, y, z) \). If we deform the solid, point \( \mathbf{r} \) will be moved to a new position and we call the new position \( \mathbf{r}' \). The deformation of this point is

\[ \mathbf{\tilde{u}} = \mathbf{r}' - \mathbf{r} \]  

(4.3)

Notice that for each point \( \mathbf{r} \), we will have a deformation vector \( \mathbf{\tilde{u}} \). So \( \mathbf{\tilde{u}} \) is a function of \( \mathbf{r} \), \( \mathbf{\tilde{u}}(\mathbf{r}) \).

\[ \mathbf{\tilde{u}}(\mathbf{r}) = u_x(\mathbf{r}) \hat{x} + u_y(\mathbf{r}) \hat{y} + u_z(\mathbf{r}) \hat{z} \]  

(4.4)

The strain tensor is defined as:
\[
\begin{pmatrix}
\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\
\varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\
\varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}
\end{pmatrix}
\]  
(4.5)

where

\[
\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}
\]

\[
\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2} \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \quad \varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2} \left( \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} \right) \quad \varepsilon_{zx} = \varepsilon_{xz} = \frac{1}{2} \left( \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right)
\]

(4.6)

This tensor is a symmetric tensor (\(e_{ab} = e_{ba}\)).

There are six independent component for a 3x3 symmetric tensor.

In the text book, the off-diagonal terms are defined without the fact 1/2. (In that case, one can NOT write these \(e_{ab}\) as a tensor).

Remark: a more precise definition of the strain tensor is

\[
\varepsilon_{xx} = \frac{\partial u_x}{\partial x} + \frac{1}{2} \frac{\partial u_x}{\partial y} \frac{\partial u_y}{\partial x} \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} + \frac{1}{2} \frac{\partial u_y}{\partial y} \frac{\partial u_y}{\partial y} \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} + \frac{1}{2} \frac{\partial u_z}{\partial z} \frac{\partial u_z}{\partial z}
\]

\[
\varepsilon_{xy} = \frac{1}{2} \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \quad \varepsilon_{yz} = \frac{1}{2} \left( \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) \quad \varepsilon_{zx} = \frac{1}{2} \left( \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right)
\]

(4.8)

(4.9)

We typically assume that the second order derivatives are much smaller than the first order derivatives, so that we can ignore them. As a result, it recovers the definition used above.

### 4.2.2. Review on vectors and tensors

**Q:** What is a vector?

**A:** Two criteria.
- Write three components in a row
- Rotational same as the coordinates. \(Rx\)

**Q:** What is a tensor?

**A:** Two criteria.
- Write nine components as a three by three matrix
- Rotation: \(RM\)

**Q:** What are symmetric and anti-symmetric tensors? Why should we care about them?

**A:** Symmetric tensor: \(M_{ij} = M_{ji}\). Anti-symmetric tensor: \(M_{ij} = -M_{ji}\). For any tensor \(M_{ij}\), we can write it as the sum of a symmetric tensor and an any symmetric tensor.

\[
M = M^S + M^A
\]

(4.10)

Here \(M\) is an arbitrary tensor, \(M^S\) is a symmetric tensor and \(M^A\) is an anti-symmetric tensor. Here,

\[
M^S = \frac{M + M^T}{2}
\]

(4.11)

\[
M^A = \frac{M - M^T}{2}
\]

(4.12)

Where \(M^T\) is the transpose matrix of the matrix \(M\).

**Q:** If I have a vector field \(\vec{u}(\vec{r})\), can I get a tensor from it?
A: Yes. The first order derivatives \( \partial_i u_j \) gives you a tensor.

\[
U = \begin{pmatrix}
\partial_i u_k & \partial_i u_y & \partial_i u_z \\
\partial_i u_k & \partial_i u_y & \partial_i u_z \\
\partial_i u_k & \partial_i u_y & \partial_i u_z
\end{pmatrix}
\]  

(4.13)

This tensor can be written as the sum of a symmetric tensor and an anti-symmetric tensor

\[
U = \begin{pmatrix}
\partial_i u_k & \frac{\partial_{ij} + \partial_{ji}}{2} & \frac{\partial_{ij} + \partial_{ji}}{2} \\
\frac{\partial_{ij} + \partial_{ji}}{2} & \partial_i u_y & \frac{\partial_{ij} + \partial_{ji}}{2} \\
\frac{\partial_{ij} + \partial_{ji}}{2} & \frac{\partial_{ij} + \partial_{ji}}{2} & \partial_i u_z
\end{pmatrix} + \begin{pmatrix}
0 & \frac{\partial_{ij} - \partial_{ji}}{2} & \frac{\partial_{ij} - \partial_{ji}}{2} \\
\frac{\partial_{ij} - \partial_{ji}}{2} & 0 & \frac{\partial_{ij} - \partial_{ji}}{2} \\
\frac{\partial_{ij} - \partial_{ji}}{2} & \frac{\partial_{ij} - \partial_{ji}}{2} & 0
\end{pmatrix}
\]  

(4.14)

The symmetric part is the strain tensor.

4.2.3. Physical meaning of the strain tensor

The strain tensor measures the change of distance between any two points A and B in a solid.

Let’s consider two nearby points A and B. Before we deform the solid, the distance between A and B is \( \vec{d} r = d x \hat{x} + d y \hat{y} + d z \hat{z} \).

After deformation, the distance between A and B changes into \( \vec{d} r' \). One can show that

\[ d r^2 = d r^2 + 2 e_{ij} d r_i d r_j + O(d r^3) \]  

(4.15)

4.2.4. Elastic energy

For a spring, the elastic energy only depends on the change of length (distance)

\[ U = \frac{1}{2} C \Delta L^2 \]  

(4.16)

For a solid, the elastic energy depends on the change of length between any two points too, which is described by the strain tensor

\[ U = \int d \vec{r} \sum_{ijkl} \frac{1}{2} C_{ijkl} \epsilon_{ij}^{(r)} \epsilon_{kl}^{(r)} \]  

(4.17)

\( C_{ijkl} \) is a rank-4 tensor, which is known as the elastic modulus tensor. Each of the component is known as an elastic stiffness Constant (or simply an elastic Constant). They measures how “hard” this solid is. A large elastic stiffness constant means that it cost more energy to deform this solid.

In other words, the solid is “hard”. Smaller elastic Constant means that the solid is “soft”.

For a rank 4 tensor, there are four indices, i, j, k and l, each of which takes three possible values x, y and z. So we have 3 x 3 x 3 x 3 = 81 different \( C_{ijkl} \) (81 elastic constants). But in reality, we just need to know 21 of them, and the value of the other 60 can be determined from these 21 elastic constants.

This is because \( e_{ij} = e_{ji} \) - we can prove that \( C_{ijkl} = C_{jikl} = C_{ijlk} \). In addition one can prove that \( C_{ijkl} = C_{klij} \) so we have

\[ C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij} \]  

(4.18)

For a spring, there is only 1 elastic constant (the spring constant). For a solid, there are 21 elastic constants.

Remark: if we choose proper axes (choosing the direction of the x, y and z axes), we can make three of the 21 elastic constants 0. So, in reality, we have only 18 independent elastic constants.

4.2.5. Another way to write down the elastic energy

Define \( \epsilon_{ij} \) with \( i = 1,2,\ldots,6 \). Here \( \epsilon_1 = e_{xx}, \epsilon_2 = e_{yy}, \epsilon_3 = e_{zz}, \epsilon_4 = e_{xy} = e_{yx}, \epsilon_5 = e_{xz} = e_{zx}, \epsilon_6 = e_{yz} = e_{zy} \)

\[ U = \int d \vec{r} \sum_{i=1}^{6} \sum_{j=1}^{6} \frac{1}{2} C_{ij} \epsilon_i^{(r)} \epsilon_j^{(r)} \]  

(4.19)

Here, one can prove that \( C_{ij} = C_{ji} \). In other words, \( C_{ij} \) is a 6x6 symmetric matrix.

For a 6x6 symmetric matrix, there are 21 components. So we have 21 elastic constants (same number as using \( C_{ijkl} \)).