6 Free Electron Fermi Gas

6.1. Electrons in a metal

6.1.1. Electrons in one atom

One electron in an atom (a hydrogen-like atom): the nucleon has charge $+Ze$, where $Z$ is the atomic number, and there is one electron moving around this nucleon

Four quantum numbers: $n, l$ and $l_z, s_z$.

Energy levels $E_n$ with $n = 1, 2, 3...$

$$E_n = -\frac{Z^2 \mu e^4}{32\pi^2 \epsilon_0^2 R^2} \frac{1}{n^2}$$  \hspace{1cm} (6.1)

where $\mu = m_e m_N / (m_e + m_N) \approx m_e$ where $m_e$ is the mass of an electron and $m_N$ is the mass of the nucleon.

$$E_n = - \frac{(13.6 \text{ eV}) Z^2}{n^2}$$  \hspace{1cm} (6.2)

For each $n$, the angular momentum quantum number $[L^2 \psi = l(l + 1) \psi]$ can take the values of $l = 0, 1, 2, 3, 4 ..., n - 1$. These states are known as the $s, p, d, f, g, ...$ states.

For each $l$, the quantum number for $L_z$ can be any integer between $-l$ and $+l$ ($l_z = -l, -l + 1, ..., l - 1, l$).

For fixed $n, l$ and $l_z$, the spin quantum number $s_z$ can be +1/2 or -1/2 (up or down).

At each $n$, there are $2n^2$ quantum states.

In a real atom

In a hydrogen-like atom, for the same $n$, all the $2n^2$ quantum number has the same energy. In a real atom, the energy level splits according to the total angular momentum quantum number $l$.

Typically, the state with lower $l$ has lower energy (not always true). Due to the rotational symmetry, states with the same $l$ has the same energy (if we ignore the spin-orbital coupling, etc.).
Many electrons at $T = 0$
Electrons are fermions. The Pauli exclusive principle requires that we can have at most 1 electron per quantum state.

At $T = 0$, to minimize the total energy, the electrons want to state at the lowest $N_e$ quantum state.

<table>
<thead>
<tr>
<th>$N_e$</th>
<th>Element</th>
<th>No. of electrons on each shell</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$H$</td>
<td>$1 , s^1$</td>
</tr>
<tr>
<td>2</td>
<td>He</td>
<td>$1 , s^2$</td>
</tr>
<tr>
<td>3</td>
<td>Li</td>
<td>$1 , s^2 , 2 , s^1$</td>
</tr>
<tr>
<td>4</td>
<td>Be</td>
<td>$1 , s^2 , 2 , s^2$</td>
</tr>
<tr>
<td>5</td>
<td>$B$</td>
<td>$1 , s^2 , 2 , s^2 , 2 , p^1$</td>
</tr>
<tr>
<td>6</td>
<td>$C$</td>
<td>$1 , s^2 , 2 , s^2 , 2 , p^2$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>10</td>
<td>Ne</td>
<td>$1 , s^2 , 2 , s^2 , 2 , p^6$</td>
</tr>
<tr>
<td>11</td>
<td>Na</td>
<td>$1 , s^2 , 2 , s^2 , 2 , p^2 , 3 , s^1$</td>
</tr>
<tr>
<td>12</td>
<td>Mg</td>
<td>$1 , s^2 , 2 , s^2 , 2 , p^2 , 3 , s^2$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
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</tr>
</tbody>
</table>

Valence electrons
Electrons in the low energy states (inner layers) are bonded tightly to the nucleon. (It costs about 1-10 eV to remove one of these electrons from an atom. 1eV is $10^4$ K. It is a very high energy cost in solid state physics.)

Electrons in high energy states (outer layers) are loosely bonded to the nucleon (easy to remove). These electrons are called the valence electrons and these energy states are called the valence shells.

Valence electrons are the electrons in a atom which can participate in the formation of chemical bonds. They are typically electrons in the outermost (or second-outer-most) shell.

6.1.2. Electrons in a metal
In a metal, an atom may lose some or all of its valence electrons and thus turns into an ion (nucleon+inner electrons). These ions form a crystal and their motions are phonons. The valence electrons are no longer bonded to nucleons. They can move freely in a crystal.

A crystal = A lattice of ions + valence electrons

Motions in a crystal = phonons + valence electrons

At low temperature, the interactions between phonons are typically very weak. So we can consider treat them as a quantum gas (a Bose gas).

In a metal, because valence electrons can move around, we can treat them as a quantum fluid (a fermion fluid). Typically, we call this fluid a Fermi liquid. It is called a Fermi liquid, instead of a Fermi gas, because the interactions between electrons (Coulomb interactions) are typically pretty strong (comparable to the kinetic energy in most metals).

A solid can be considered as the mixture of two type of fluids:

A Bose gas of phonons + A Fermi liquid of electrons

6.1.3. Free electrons in 1D at zero temperature $T = 0$
Here, we start from the simplest situation: a free Fermi gas

The word “free” here means two things.

1. We ignore interactions between electrons
2. We ignore interactions between electrons and ions (nucleons)

Within this approximation, electrons are free particles. So their Hamiltonian is (for a 1D system)

$$ H = \frac{p^2}{2 \, m} = \frac{1}{2 \, m} \left( -i \, \hbar \, \frac{d}{dx} \right)^2 = -\frac{\hbar^2}{2 \, m} \frac{d^2}{dx^2} $$

One electron in 1D
The Schrodinger equation is
\[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi_n(x) = \epsilon_n \psi_n(x) \]  \hspace{2cm} (6.7)

The solutions of this equation are plane waves
\[\psi_n(x) = A \exp(i k_n x)\]  \hspace{2cm} (6.8)

The eigen-energy \(\epsilon_n\) is
\[\epsilon_n = \frac{\hbar^2 k_n^2}{2m}\]  \hspace{2cm} (6.9)

For a 1D system with length \(L\) and periodic boundary conditions, \(\psi_n(x) = \psi_n(x + L)\), we know that the wavevector (momentum) is quantized
\[k_n L = 2n \pi\]  \hspace{2cm} (6.10)

with \(n\) being an integer, \(n = \ldots, -2, -1, 0, 1, 2, \ldots\) Or say,
\[k_n = 2n \frac{\pi}{L}\]  \hspace{2cm} (6.11)

The eigen-energy here is also quantized
\[\epsilon_n = \frac{\hbar^2}{2m} \left( \frac{2n \pi}{L} \right)^2\]  \hspace{2cm} (6.12)

For a large system \(L \rightarrow \infty\), the energy and momentum become continuous variables.

**Many electrons in 1D at \(T = 0\)**

At \(T = 0\), the electrons want to stay in the lowest energy states to minimize the energy. But electrons are fermions, which means that they cannot all just go to the lowest energy state \(\epsilon_0\). Due to the Pauli exclusive principle, one quantum state can host at most one electron. These mean that for each \(n\), we can have at most two electrons (one with spin up and one with spin down).

If we have \(N\) electrons, at \(T = 0\), the electron occupies the lowest \(N/2\) states. The energy of the highest filled state is known as the Fermi energy \(\epsilon_F\). The momentum of this state is known as the Fermi momentum \(P_F\). The wavevector of this state is known as the Fermi wavevector \(k_F\). Obviously,

\[P_F = \hbar k_F \quad \text{and} \quad \epsilon_F = \frac{P_F^2}{2m} = \frac{\hbar^2 k_F^2}{2m}\]  \hspace{2cm} (6.13)

**For a large system \((L \rightarrow \infty), \epsilon_F = ?\)**

The number of quantum state between \(-P_F\) to \(P_F\) is (the factor 2 comes from the spin degrees of freedom):

\[2 \int_{-P_F}^{P_F} \frac{dP}{2\pi \hbar} = 2 \frac{L}{\pi \hbar} \int_{-P_F}^{P_F} dP = 2 \frac{P_F}{\pi \hbar} = 2 \frac{L}{\pi \hbar} P_F\]  \hspace{2cm} (6.14)

Number of electrons in these states
\[N_e = 2 \frac{L}{\pi \hbar} P_F\]  \hspace{2cm} (6.15)

So
\[P_F = \frac{\hbar N_e}{2L} = \frac{\pi \hbar}{2} \left( \frac{N_e}{L} \right)\]  \hspace{2cm} (6.16)

Here \(N_e / L\) is the density of electrons (number of electrons per length).

\[\epsilon_F = \frac{P_F^2}{2m} = \frac{\pi^2 \hbar^2}{8m} \left( \frac{N_e}{L} \right)^2\]  \hspace{2cm} (6.17)

The bottom line: \(P_F, \epsilon_F\) and \(k_F\) are determined by the density of electrons.
6.1.4. Free electrons in 1D at finite temperature

Distribution function

The distribution function \( f(\epsilon) \) measures the average number of electrons on a quantum state with energy \( \epsilon \).

At zero temperature, the number of electrons on a quantum state is 1 if \( \epsilon < \epsilon_F \) and 0 if \( \epsilon > \epsilon_F \). In other words, this function is a step function

\[
f(\epsilon) = \begin{cases} 
1 & \epsilon < \epsilon_F \\
0 & \epsilon > \epsilon_F 
\end{cases}
\]  

(6.18)

At finite \( T \), the electron number on a quantum state is NOT fixed. Sometimes this state is occupied (1 electron) and sometimes it is empty (0 electron). So the average electron number is some number between 0 and 1.

\[
0 < f(\epsilon) < 1 \text{ at finite } T
\]  

(6.19)

Fermi distribution:

For non-interacting fermions, at finite temperature, the distribution function takes this form

\[
f(\epsilon) = \frac{1}{\exp\left(\frac{\epsilon - \mu}{k_B T}\right) + 1}
\]  

(6.20)

where is known as the Fermi-Dirac distribution.

Let’s compare it with the Planck distribution (for phonons) we learned in the previous chapter.

\[
f_{\text{phonon}}(\epsilon) = \frac{1}{\exp\left(\frac{\epsilon}{k_B T}\right) - 1}
\]  

(6.21)

We found two differences: (1) -1 turns into +1 and (2) we have an extra parameter \( \mu \) for electrons.

The \( \pm 1 \) is determined by the nature of the particles. For bosons (photons, phonons, etc.), it is \( -1 \). For fermions (electrons, protons), it is always \( +1 \).

The parameter \( \mu \) is called the chemical potential. It is a very important parameter in thermodynamics and statistical physics (it is as important as the concept of temperature). It controls the density of particles in a system. Here, for a Fermi gas, we can consider it as a generalization of \( \epsilon_F \) at finite \( T \). In fact, we will show below that when \( T \to 0 \), \( \mu \to \epsilon_F \).

How to determine \( \mu \)?

The chemical potential \( \mu \) can be determined by counting the total number of electrons

\[
\frac{N_e}{L} = 2 \int_{-\infty}^{\infty} \frac{dP}{2\pi \hbar / L} f(\epsilon) = \frac{L}{\pi \hbar} \int_{-\infty}^{\infty} dP \frac{1}{\exp\left(\frac{\epsilon - \mu}{k_B T}\right) + 1}
\]  

(6.22)

Divided by \( L \) on both sides:

\[
\frac{N_e}{L} = \frac{1}{\pi \hbar} \int_{-\infty}^{\infty} dP \frac{1}{\exp\left(\frac{\epsilon - \mu}{k_B T}\right) + 1}
\]  

(6.23)

If we know the density of electrons \( N_e / L \), we can solve this equation to find \( \mu \). In general, \( \mu \) depends on \( T \) and the electron density. As \( T \to 0 \), \( \mu \to \epsilon_F \).

Some properties of \( f(\epsilon) \)

(a) \( f(\epsilon) \) turns into a step function at \( T \to 0 \).

At low \( T \), \( \frac{\epsilon - \mu}{k_B T} \) becomes very large. Depending on the sign of \( \epsilon - \mu \)

\[
\lim_{T \to 0} \frac{\epsilon - \mu}{k_B T} = \begin{cases} 
-\infty & \epsilon < \mu \\
+\infty & \epsilon > \mu 
\end{cases}
\]  

(6.24)

Therefore,
\begin{align}
\text{Lim}_{T \to 0} f(\epsilon) &= \text{Lim}_{T \to 0} \frac{1}{\exp\left(\frac{\epsilon - \mu}{kT}\right) + 1} = \left\{ \begin{array}{ll}
\frac{1}{\exp\left(-\infty\right)} + 1 = \frac{1}{0} = 1 & \epsilon < \mu \\
\frac{1}{\exp\left(\infty\right)} + 1 = \frac{1}{\infty} = 1 & \epsilon > \mu
\end{array} \right.
\end{align}

(6.25)

This result recovers the zero temperature limit we studied above. And it is easy to notice that the chemical potential \( \mu \) plays the role of \( \epsilon_f \) here. In other words, \( \mu = \epsilon_f \) at \( T = 0 \).

\textbf{(b) At any} \( T \), \( f(\mu) = \frac{1}{2} \)

\[
f(\mu) = \frac{1}{\exp\left(\frac{\mu - \mu}{kT}\right) + 1} = \frac{1}{\exp(0) + 1} = \frac{1}{2}
\]

(6.26)

\textbf{(c) At} \( T < \mu / k_b \), \( f(\epsilon) \) \text{ is close to a step function. Only the part with} \( \epsilon - \mu \) \text{ shows strong deviation from the step function} \( f_{\text{Fermi}}(\epsilon) \)

\begin{align}
\text{Plot} \left\{ \frac{1}{\exp\left[\frac{\epsilon - \mu}{kT}\right] + 1} / \ T \to 0.0000001, \quad \frac{1}{\exp\left[\frac{\epsilon - 1 - \mu}{kT}\right] + 1} / \ T \to .01, \quad \frac{1}{\exp\left[\frac{\epsilon - 1}{kT}\right] + 1} / \ T \to .1, \quad \frac{1}{\exp\left[\frac{\epsilon - 1}{kT}\right] + 1} / \ T \to 1, \quad \frac{1}{\exp\left[\frac{\epsilon - 1}{kT}\right] + 1} / \ T \to 100 \right\},
\end{align}

\{\epsilon, 0, 2\}, \text{PlotStyle} \to \text{Thick}, \text{AxesLabel} \to \{"\epsilon/\mu", "f(\epsilon)"\}, \text{LabelStyle} \to \text{Medium},

\text{PlotLegends} \to \{"T=0", "T=1/100 \ k_b", "T=1/10 \ k_b", "T=\mu \ k_b", "T=100 \ \mu \ k_b"\}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fepsilon.png}
\caption{Graph showing the function \( f(\epsilon) \) for different temperatures.}
\end{figure}

6.1.5. \textbf{Free electrons in 3D at zero temperature}

For free electrons, the Hamiltonian in 3D is

\[
H = \frac{p^2}{2m} - \frac{p_x^2 + p_y^2 + p_z^2}{2m} = \frac{1}{2m} \left(-i \hbar \frac{\partial}{\partial x}\right)^2 + \left(-i \hbar \frac{\partial}{\partial y}\right)^2 + \left(-i \hbar \frac{\partial}{\partial z}\right)^2 = -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)
\]

(6.27)

\textbf{One electron in 3D}

The Schrödinger equation is

\[
-\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \psi_k(x, y, z) = \epsilon \psi_k(x, y, z)
\]

(6.28)

The solutions for this equation are 3D plane waves

\[
\psi_k(x, y, z) = \mathcal{A} \exp[i \left(k_x x + k_y y + k_z z\right)]
\]

(6.29)
Considering periodic boundary conditions
\[
\psi_k(x + L_x, y, z) = \psi_k(x, y, z)
\]
\[
\psi_k(x + L_y, y, z) = \psi_k(x, y, z)
\]
\[
\psi_k(x + L_z, y, z) = \psi_k(x, y, z)
\]
(6.30)
(6.31)
(6.32)

We find that the momenta are quantized
\[
k_x = \frac{2l \pi}{L_x} \quad \text{and} \quad k_y = \frac{2m \pi}{L_y} \quad \text{and} \quad k_z = \frac{2n \pi}{L_z}
\]
(6.33)

where \(l, m, n\) are integers.

The eigen-energy is also quantized:
\[
\epsilon_n = \frac{\hbar^2}{2m} \left( k_x^2 + k_y^2 + k_z^2 \right) = \frac{\hbar^2}{2m} \left( \frac{4\pi^2}{L_x^2} + \frac{4\pi^2}{L_y^2} m^2 + \frac{4\pi^2}{L_z^2} n^2 \right)
\]
(6.34)

For a very large system \((L \to \infty)\), the discrete energy and momenta turn into continuous variables.

**Many electrons in 3D at \(T = 0\)**

At \(T = 0\), the electrons occupy the lowest \(N\) quantum states to save energy. In other words, the quantum states with \(\epsilon \leq \epsilon_F\) are occupied, while states with \(\epsilon > \epsilon_F\) are empty.

Since \(\epsilon = \frac{\hbar^2 k^2}{2m}\), this means that states with momentum \(k \leq k_F\) are occupied and states with \(k > k_F\) are empty. Here \(k_F = \sqrt{2\epsilon_F m / \hbar}\). In other words, in the \(k\)-space, the occupied states form a sphere with radius \(k_F\). This sphere is known as the Fermi sphere (or the Fermi sea).

Q: \(\epsilon_F = \epsilon\) \((k_F = k)\)?

A: It is determined by the density of electrons.

The total volume of the Fermi sphere is \(\frac{4\pi}{3} k_F^3\). Each quantum state occupies the volume \((2\pi)^3 / V\), which comes from the uncertainty relation. So the total number of quantum states is
\[
\frac{\frac{4\pi}{3} k_F^3}{(2\pi)^3 / V} = \frac{V k_F^3}{6\pi^2}
\]
(6.35)

There are two electrons per state (because we have electrons with spin up and spin down), so the total number of electrons is
\[
N = 2 \times \frac{V k_F^3}{6\pi^2} = \frac{V k_F^3}{3\pi^2}
\]
(6.36)

So
\[
k_F = \left( \frac{3\pi^2 N}{V} \right)^{1/3}
\]
(6.37)

\(k_F\) is determined by the electron density \(N / V\).

The Fermi energy
\[
\epsilon_F = \frac{\hbar^2}{2m} k_F^2 = \frac{\hbar^2}{2m} \left( \frac{3\pi^2}{V} \right)^{2/3} N^{1/3}
\]
(6.38)

The Fermi velocity
\[
\nu_F = \frac{d\epsilon}{d k} \big|_{k = k_F} = \frac{\hbar}{m} k_F = \frac{\hbar}{m} \left( \frac{3\pi^2}{V} \right)^{1/3} N^{1/3}
\]
(6.39)

Here, the formula \(\nu = \frac{d\epsilon}{d k}\) is nothing but the definition of the group velocity. Notice that \(\epsilon / \hbar = \omega\) and \(\nu = d\omega / dk\) is precisely the definition of the group velocity.
Another equivalent definition of the Fermi velocity is

$$v_F \equiv \frac{P_F}{m}$$  \hspace{1cm} (6.40)$$

If the energy $\epsilon$ is a quadratic function of $k$ ($\epsilon = \hbar^2 k^2 / 2m$), these two definitions are identical. If $\epsilon$ is NOT a quadratic function of $k$ (which could happen as will be discussed in the next chapter), we use the first definition $v_F = \frac{d\epsilon}{\hbar \, dk} \big|_{k = k_F}$.

The bottom line: the first definition is more generic. The second one ($p_F / m$) only works for $\epsilon = \hbar^2 k^2 / 2m$.

### 6.1.6. Conductivity

Under a static electric field $\vec{E}$, electrons feel a constant force.

$$\vec{F} = -e \vec{E}$$  \hspace{1cm} (6.41)$$

The second Law of Newton tell us that

$$\vec{F} = \frac{d\vec{P}}{dt}$$  \hspace{1cm} (6.42)$$

If we have a constant force, $\vec{P}$ increases linearly as a function of $t$

$$\vec{P}(t) = \vec{P}(t = 0) + \vec{F} \, t$$  \hspace{1cm} (6.43)$$

In quantum mechanics, $\vec{P} = \hbar \vec{k}$, so we have

$$\vec{k}(t) = \vec{k}(t = 0) + \frac{\vec{F}}{\hbar} \, t = \vec{k}(t = 0) - e \frac{\vec{E}}{\hbar} \, t$$  \hspace{1cm} (6.44)$$

Here, if we turn on the $E$ field at $t = 0$, at $t \leq 0$, the Fermi sea is a sphere centered at $k = 0$. At $t > 0$, the Fermi sphere is centered at $\partial k$ ($\equiv \frac{-e \vec{E}}{\hbar}$), because the wavevector of every electron is shifted by this amount at time $t$.

For free electrons in a perfect crystal, $\delta k(t) \to \infty$ as $t \to \infty$. However, in a real solid, $\delta k(t)$ will NOT diverge, because there are collisions between electrons, between electrons and phonons, and between electrons and impurities. These scatterings will reduce the momentums towards $\delta k = 0$. They will introduce viscosity to the Fermi liquid, which reduces the velocity and momentum of the liquid. Eventually, the scatterings and the $E$-field will balance each other and $\delta k$ will turn to a fixed value.

The total force can be written as

$$\vec{F} = -e \vec{E} - \frac{m}{\tau} \vec{v} = -e \vec{E} - \frac{\vec{p}}{\tau}$$  \hspace{1cm} (6.45)$$

Here the first term is the electric force and the second term is a viscosity term, which is proportional to $-\vec{v}$, reducing the speed. The coefficients $m$ and $\tau$ are the electron mass and the collision time (average time between two collisions).

Using the second law of Newton:

$$\frac{d\vec{p}}{dt} = -e \vec{E} - \frac{\vec{p} - \vec{p}_0}{\tau}$$  \hspace{1cm} (6.46)$$

where $\vec{p}_0$ is the momentum before we turn on the electric field and we assume the initial condition $\vec{p}(t = 0) = \vec{p}_0$.

It is easy to find that the solution to this equation is

$$\vec{p} = \vec{p}_0 - e \vec{E} \, \tau \, (1 - e^{-t/\tau})$$  \hspace{1cm} (6.47)$$

This means that the change of momentum

$$\delta \vec{p} = \vec{p} - \vec{p}_0 = -e \vec{E} \, \tau \, (1 - e^{-t/\tau})$$  \hspace{1cm} (6.48)$$

In the static limit ($t \to \infty$),
The change in wavevector
\[ \delta k = -\frac{e \vec{E}}{\hbar} \tau \]

This approximation is known as the Drude approximation (the Drude model).

**The classical theory (ignore the fact that electrons are fermions)**

The total electric current is
\[ j = -e n v = -e n \frac{h \delta k}{m} = \frac{e^2 n \tau}{m} \vec{E} \]

This is the Ohm’s Law
\[ j = \sigma \vec{E} \]

where the conductivity
\[ \sigma = \frac{e^2 n \tau}{m} \]

**The quantum theory (taken into account the Pauli exclusive principle)**

Assuming that the E field is along the z direction. The total velocity along z is
\[ j = -(e n v) = -\left(\frac{e}{V} N v\right) = -\frac{e}{V} \frac{2 \pi^2}{(2 \pi)^3 / V} \int_0^{2 \pi} \int_0^\pi d \phi \int_0^\pi d \theta k F^2 \delta k \cos \theta \times v F \cos \theta = \]

\[ -e \frac{k F^2 v F \delta k}{2 \pi^2} \int_0^\pi d \theta \cos^2 \theta = -e \frac{k F^2 v F \delta k}{4 \pi^2} = -e \frac{k F^2 v F}{4 \pi^2} \left( -\frac{e \vec{E}}{h} \tau \right) = e^2 \frac{k F^2 v F}{4 \pi^2 h} \tau E = \frac{e^2}{4 \pi^2 m} \tau E = \frac{e^2}{4 \pi^2 m} \tau E \]

Because
\[ k F = \left( \frac{3 \pi^2 N}{V} \right)^{1/3} \]

\[ j = \frac{e^2 k F^3}{m 4 \pi^2} \tau E = \frac{e^2 3 \pi^2 N}{m 4 \pi^2} \tau E = \frac{3 e^2 n \tau}{4 m} \]

So
\[ \sigma = \frac{3 e^2 n \tau}{4 m} \]

Typically, we absorb the extra 3/4 factor into the definition of \( \tau \), so the conductivity turns back to its classical form
\[ \sigma = \frac{e^2 n \tau}{m} \]

which is identical to the classical formula shown above.

Sometimes, we use the mean free path \( l = v F \tau \) in the formula, instead of \( \tau \). The physical meaning of the mean free path is the average distance that an electron travels between two collisions. Using \( l \), we have
\[ \sigma = \frac{e^2 n l}{m v F} \]

The resistivity \( \rho \) is
\[ \rho = \frac{m}{e^2 n \tau} \]  \hspace{1cm} (6.60)

**Collision time \(\tau\)**

The collision time \(\tau\) comes from

- collisions between electrons \(\tau_{ee}\)
- collisions between electrons and phonons: \(\tau_{ep}\)
- collisions between electrons and impurities: \(\tau_i\)

If we assume that different scatterings are independent, the total \(\tau\) satisfies

\[ \frac{1}{\tau} = \frac{1}{\tau_{ee}} + \frac{1}{\tau_{ep}} + \frac{1}{\tau_i} \]  \hspace{1cm} (6.61)

Notice that

\[ \rho = \frac{m}{e^2 n \tau} \]  \hspace{1cm} (6.62)

So

\[ \rho = \frac{m}{e^2 n} \left( \frac{1}{\tau_{ee}} + \frac{1}{\tau_{ep}} + \frac{1}{\tau_i} \right) = \frac{m}{e^2 n \tau_{ee}} + \frac{m}{e^2 n \tau_{ep}} + \frac{m}{e^2 n \tau_i} \]  \hspace{1cm} (6.63)

If we define

\[ \rho_{ee} = \frac{m}{e^2 n} \frac{1}{\tau_{ee}} \quad \text{and} \quad \rho_{ep} = \frac{m}{e^2 n} \frac{1}{\tau_{ep}} \quad \text{and} \quad \rho_i = \frac{m}{e^2 n} \frac{1}{\tau_i} \]  \hspace{1cm} (6.64)

so we found

\[ \rho = \rho_{ee} + \rho_{ep} + \rho_i \]  \hspace{1cm} (6.65)

**Resistivity in a real metal at finite temperature**

In a real metal at finite temperature, the approximation used above are still valid, but the collision time \(\tau\) will show temperature-dependence. As a result, \(\rho\) is a function of \(T\). Typically, \(\rho\) decreases as \(T\) goes down. At \(T = 0\), \(\rho\) goes to a finite value \(\rho(T = 0)\), which is known as the residue resistivity.

At low \(T\), \(\rho_{ee}\) and \(\rho_{ep}\) decreases to 0 as \(T\) is reduced down to 0, while \(\rho_i\) is \(T\) independent. Very typically, \(\rho_{ee} \propto T^2\), \(\rho_{ep} \propto T^5\) and \(\rho_i \propto \text{constant}\)

\[ \rho = \rho_i + A_{ee} T^2 + A_{ep} T^5 \]  \hspace{1cm} (6.66)

At low temperature, \(\rho_i >> \rho_{ee} >> \rho_{ep}\), so impurity scattering is the dominate contribution for \(\rho\). At high temperature, \(\rho\) is typically a linear function of \(T\)

\[ \rho = A T \]  \hspace{1cm} (6.67)

This is because the number of phonons is proportional to \(T\) at high temperature.

\[ n = \frac{1}{\exp\left( \frac{\epsilon}{k_b T} \right) - 1} \approx \frac{1}{\left(1 + \frac{\epsilon}{k_b T} + O(T^{-2}) \right) - 1} = \frac{k_B T}{\epsilon} \]  \hspace{1cm} (6.68)

**6.1.7. the Hall effect (part I)**

Consider a thin sample (electrons moving in 2D). We apply a magnetic field perpendicular to the plane (in the \(z\) direction)

\[ \vec{F} = -e(E + \vec{v} \times \vec{B}) \]  \hspace{1cm} (6.69)

The second term is the Lorentz force, which is perpendicular to the velocity. If we pass a current along the \(x\) direction. The electrons will feel a Lorentz force along the \(y\) direction and thus positive (negative) charge will accumulate along the top and bottom edge of the sample. This charge accumulation will induce a electric field along \(y\), perpendicular to the current. If we have a static current, this electric field will balance the
Lorentz force.

\[-e E_y \hat{e}_y = e \vec{v} \times \vec{B} = -e v_x B \hat{e}_y\]  

(6.70)

So

\[E_y = v_x B\]  

(6.71)

The current along x is

\[j_x = -e n v_x\]  

(6.72)

So the ratio between \(E_y\) and \(j_x\), which is called the Hall resistance, is

\[R_{xy} = \frac{E_y}{j_x} = \frac{v_x B}{-e n v_x} = \frac{1}{e n} B\]  

(6.73)

The ratio between \(R_{xy}\) and \(B\) is known as the Hall coefficient

\[R_H = \frac{R_{xy}}{B} = \frac{1}{e n}\]  

(6.74)

\(R_H\) is proportional to \(1/n\) (inverse of the electron density), this measurement is the standard technique to determine the density of electrons in a material.

\[n = -\frac{1}{e R_H}\]  

(6.75)

Notice that the density \(n\) here is 2D electron density (number of electron per area)

\[n = n_{2D} = \frac{N_e}{A}\]  

(6.76)

For a 3D material, the 3D electron density is

\[n = n_{3D} = \frac{N_e}{V} = \frac{N_e}{A d} = n_{2D}/d\]  

(6.77)

where \(d\) is the thickness of the sample.

### 6.1.8. the Hall effect (part II with scatterings)

In the calculations above, we didn’t consider the electron collisions. If we treat the collisions using the Drude approximation (assuming that the collisions are described by a single parameter \(\tau\), the collision time), the force on an electron is

\[\vec{F} = -e \left(\vec{E} + \vec{v} \times \vec{B}\right) - \frac{m}{\tau} \vec{v}\]  

(6.78)

Using the second law of Newton, we find that

\[m \frac{d \vec{v}}{dt} = -e \left(\vec{E} + \vec{v} \times \vec{B}\right) - \frac{m}{\tau} \vec{v}\]  

(6.79)

So we have two equations of motion

\[m \frac{d v_x}{dt} + e v_x B = -e E_x\]  

(6.80)

\[m \frac{d v_y}{dt} - e v_y B = -e E_y\]  

(6.81)

Using the initial condition \(v_x = v_y = 0\) at \(t = 0\), the solution of these differential equations is

\[v_x = -\frac{e}{m^2 + B^2 e^2 \tau^2} \left[(-E, m + B e E_y \tau) \cos \frac{B e \tau}{m} + (E, m + B e E_y \tau) \sin \frac{B e \tau}{m}\right] \]  

(6.82)
\[ v_y = -\frac{e \tau (E_y m + Be E_x \tau)}{m^2 + B^2 e^2 \tau^2} + e^{i \frac{m}{m}} \left[ \frac{e \tau \left( \frac{E_x m + Be E_y \tau}{m} \cos \frac{B e \tau}{m} + \frac{E_x m - Be E_y \tau}{m} \sin \frac{B e \tau}{m} \right)}{m^2 + B^2 e^2 \tau^2} \right] \]

(6.83)

We can define the cyclotron frequency \( \omega_c = e B / m \), so that

\[ v_x = -\frac{e \tau \left( E_x - \omega_c \tau E_y \right)}{m} \frac{1}{1 + \omega_c^2 \tau^2} + \frac{e \tau \left( -E_x + \omega_c \tau E_y \right)}{m} \frac{1}{1 + \omega_c^2 \tau^2} \]

(6.84)

\[ v_y = -\frac{e \tau \left( E_y + \omega_c \tau E_x \right)}{m} \frac{1}{1 + \omega_c^2 \tau^2} + \frac{e \tau \left( E_y - \omega_c \tau E_x \right)}{m} \frac{1}{1 + \omega_c^2 \tau^2} \]

(6.85)

The static solution at \( t \to \infty \) is

\[ v_x = -\frac{e \tau \left( E_x - \omega_c \tau E_y \right)}{m} \frac{1}{1 + \omega_c^2 \tau^2} \]

(6.86)

\[ v_y = -\frac{e \tau \left( E_y + \omega_c \tau E_x \right)}{m} \frac{1}{1 + \omega_c^2 \tau^2} \]

(6.87)

If the current is along \( x \) \( (v_y = 0) \), we find that

\[ E_y = -\omega_c \tau E_x \]

(6.88)

So

\[ v_x = -\frac{e \tau \left( E_x - \omega_c \tau E_y \right)}{m} \frac{1}{1 + \omega_c^2 \tau^2} = -\frac{e \tau \left( E_x + \omega_c^2 \tau^2 E_x \right)}{m} \frac{1}{1 + \omega_c^2 \tau^2} = -\frac{E_x}{m} \]

(6.89)

Therefore,

\[ R_{xy} = \frac{E_y}{i_x} = \frac{E_x}{-ne \nu x} = \frac{-\omega_c \tau E_x}{-ne \nu x} = \frac{1}{m \omega_c} \frac{1}{m - Be} \frac{1}{m} = \frac{1}{m} \frac{1}{ne} \]

(6.90)

And thus

\[ R_{ji} = \frac{R_{xy}}{B} = \frac{1}{ne} \]

(6.91)

### 6.1.9. Free electrons in 3D at finite temperature

Number of electrons on a quantum state with energy \( \epsilon \) (the Fermi-Dirac distribution function)

\[ f(\epsilon) = \frac{1}{\exp \left( \frac{\epsilon - \mu}{k_B T} \right) + 1} \]

(6.92)

Notice that \( 0 \leq f(\epsilon) \leq 1 \), which agrees with the Pauli exclusive principle.

#### Density of states (DOS)

How many states do we have in the window of \( \epsilon \to \epsilon + d \epsilon \)?

\[ D(\epsilon) d \epsilon = ? \]

(6.93)

To answer this question, let’s consider the total number of electrons

\[ N = \int D(\epsilon) d \epsilon \]

(6.94)

Total number of electrons can also be written as
\[
N = 2 \times \int \int_{k \leq \sqrt{V}} \frac{d^4 k}{(2 \pi)^3} = 2 \sqrt{\frac{4 \pi k^2}{8 \pi^3}} = \frac{V}{\pi^2} \int_0^\infty \frac{d k^2}{d k} d k = \frac{V}{\pi^2} \int_0^\infty \frac{d k}{d \epsilon} d \epsilon
\]

We know that
\[
\frac{d \epsilon}{d k} = \frac{h^2 k^2}{2 m} \Rightarrow \frac{d \epsilon}{d k} = \frac{h^2}{m}
\]

So
\[
N = \frac{V}{\pi^2} \int_0^\infty \frac{d k^2}{d \epsilon} d \epsilon = \frac{V}{\pi^2} \int_0^\infty \frac{d \epsilon}{\sqrt{V}} = \frac{m V}{\pi^2} \int_0^\infty k d \epsilon
\]

Because \( \epsilon = h^2 k^2 / 2 m, k = \sqrt{2 m \epsilon} / h \)
\[
N = \frac{m V}{\pi^2 h^2} \int_0^\infty \frac{d \epsilon}{\sqrt{2 m \epsilon}} / h d \epsilon = \int_0^\infty \sqrt{2 m \epsilon} / 2 \pi^2 h^3 - \sqrt{\epsilon} d \epsilon = \int_0^\infty V \left( \frac{2 m \epsilon}{2 \pi^2 h^3} \right)^{3/2} \sqrt{\epsilon} d \epsilon = \int_0^\infty D(\epsilon) d \epsilon
\]

So
\[
D(\epsilon) = \left[ \frac{V}{2 \pi^2 h^2} \right]^{3/2} \sqrt{\epsilon}
\]

Another way to get \( D(\epsilon) \) is
\[
D(\epsilon) = \frac{d N}{d \epsilon}
\]

Because
\[
N = \frac{V k^3}{3 \pi^2} = \frac{V (2 m \epsilon)^{3/2}}{3 \pi^2 h^3}
\]
\[
D(\epsilon) = \frac{d N}{d \epsilon} = \frac{V (2 m \epsilon)^{3/2}}{2 \pi^2 h^3} \sqrt{\epsilon} = \frac{V}{2 \pi^2} \left( \frac{2 m \epsilon}{h^3} \right)^{3/2} \sqrt{\epsilon}
\]

Here, we can also prove that \( D(\epsilon) = 3 N / 2 \epsilon \),
\[
D(\epsilon) = \frac{d N}{d \epsilon} = \frac{3 V (2 m \epsilon)^{3/2}}{2 \pi^2 h^3} \sqrt{\epsilon} = \frac{3 V (2 m \epsilon)^{3/2}}{2 \pi^2 h^3} = \frac{3 N}{2 \epsilon}
\]

**Total energy**
\[
U(T) = \int_0^\infty \epsilon \ f(\epsilon) \ D(\epsilon) d \epsilon
\]

**Heat capacity**
\[
C_V = \frac{\partial U}{\partial T} \bigg|_V = \int_0^\infty \epsilon \ f(\epsilon) \ \frac{d}{dT} D(\epsilon) d \epsilon
\]

First we prove that \( C_V = \int_0^\infty (\epsilon - \mu) \ \frac{d f(\epsilon)}{dT} D(\epsilon) d \epsilon \)

It is easy to prove that
\[
\int_0^\infty \frac{d f(\epsilon)}{d T} D(\epsilon) d \epsilon = \frac{d}{dT} \int_0^\infty f(\epsilon) D(\epsilon) d \epsilon = \frac{d N}{d T} = 0
\]

(because the number of electrons are fixed, so \( \frac{d N}{dT} = 0 \))

Therefore,
we know that
\[ C_V = \frac{\partial U}{\partial T} \bigg|_T = \int_0^\infty \epsilon \frac{d f(\epsilon)}{d T} D(\epsilon) d\epsilon = \int_0^\infty \epsilon \frac{d f(\epsilon)}{d T} D(\epsilon) d\epsilon - \mu \times (0) = \int_0^\infty \epsilon \frac{d f(\epsilon)}{d T} D(\epsilon) d\epsilon - \mu \times \int_0^\infty \frac{d f(\epsilon)}{d T} D(\epsilon) d\epsilon = \int_0^\infty (\epsilon - \mu) \frac{d f(\epsilon)}{d T} D(\epsilon) d\epsilon \] (6.107)

We know that
\[ \frac{d f(\epsilon)}{d T} = \frac{d}{d T} \frac{1}{\exp\left(\frac{\epsilon - \mu}{k_B T}\right) + 1} = \frac{1}{\exp\left(\frac{\epsilon - \mu}{k_B T}\right) + 1}^2 \exp\left(\frac{\epsilon - \mu}{k_B T}\right) \frac{\epsilon - \mu}{k_B T} \] (6.108)

\[ \frac{d f(\epsilon)}{d \epsilon} = \frac{d}{d \epsilon} \frac{1}{\exp\left(\frac{\epsilon - \mu}{k_B T}\right) + 1} = -\frac{1}{\exp\left(\frac{\epsilon - \mu}{k_B T}\right) + 1}^2 \exp\left(\frac{\epsilon - \mu}{k_B T}\right) \frac{1}{k_B T} \] (6.109)

so
\[ \frac{d f(\epsilon)}{d T} = -\frac{d f(\epsilon)}{d \epsilon} \frac{\epsilon - \mu}{T} \] (6.110)

As a result,
\[ C_V = -\int_0^\infty (\epsilon - \mu) \frac{d f(\epsilon)}{d \epsilon} \frac{\epsilon - \mu}{T} D(\epsilon) d\epsilon = -\int_0^\infty (\epsilon - \mu)^2 \frac{d f(\epsilon)}{d \epsilon} D(\epsilon) d\epsilon \] (6.111)

At \( T << \epsilon_F / k_B \), \( \frac{d f(\epsilon)}{d \epsilon} \) has a very high peak at \( \epsilon = \epsilon_F \). So we only need to consider the integral near \( \epsilon = \epsilon_F \). Therefore, we can set \( D(\epsilon) \approx D(\epsilon_F) \).

This approximation is very good for most metals, where \( \epsilon_F / k_B \approx 100 \ T \) or larger. So we have
\[ C_V = -D(\epsilon_F) \int_0^{\epsilon_F} (\epsilon - \mu)^2 \frac{d f(\epsilon)}{d \epsilon} d\epsilon = -D(\epsilon_F) \int_0^{\epsilon_F} (\epsilon - \mu)^2 \frac{d f(\epsilon)}{T} d\epsilon = \frac{D(\epsilon_F)}{T} \int_0^{\epsilon_F} (\epsilon - \mu)^2 d\epsilon \] (6.112)
Notice that

\[ k_B T \ln(f^{-1} - 1) = \epsilon - \mu \]  \hspace{1cm} (6.113)

\[ C_V = \frac{\Delta (\epsilon_F)}{T} \int_0^\epsilon (\epsilon - \mu)^2 \, d\epsilon = \frac{D(\epsilon_F)}{T} \int_0^\epsilon \left[ k_B \ln(f^{-1} - 1) \right]^2 \, d\epsilon = D(\epsilon_F) k_B^2 \frac{\epsilon^2}{3} \] \hspace{1cm} (6.114)

So \( C_V \) is a linear function of \( T \) at \( T << \epsilon_F/k_B \).

We know that \( D(\epsilon_F) = \frac{3N}{2\epsilon_F} \), so

\[ C_V = \frac{\pi^2}{3} D(\epsilon_F) k_B^2 \frac{T}{\epsilon_F} = \frac{\pi^2}{3} \frac{3N}{2\epsilon_F} k_B^2 T = \frac{\pi^2}{2} N k_B \frac{k_B T}{\epsilon_F} \] \hspace{1cm} (6.115)

Define Fermi temperature \( T_F = \epsilon_F/k_B \)

\[ C_V = \frac{\pi^2}{2} N k_B \frac{k_B T}{\epsilon_F} = \frac{\pi^2}{2} N k_B \frac{T}{T_F} \] \hspace{1cm} (6.116)

**Heat capacity in a real metal**

In a real metal, the heat capacity takes this form at low \( T \)

\[ C_V = \gamma T + A T^3 \] \hspace{1cm} (6.117)

Here, the first term comes from electrons and the second one comes from acoustic phonons. At very low temperature, \( \gamma T >> A T^3 \), so the electron contribution dominates the heat capacity.
In experiments, one can plot the curve $C_V/T$ vs $T^2$. This curve is a linear function

$$ y = \gamma + Ax $$

(6.118)

We can determine the slope $A$ and the intercept $\gamma$ by fitting the experimental data.

**Thermal effective mass**

According to the theory of free electron gases, the coefficient $\gamma$ is

$$ \gamma = \frac{\pi^2 N k_B^2}{2 \epsilon_F} $$

(6.119)

Notice that

$$ \epsilon_F = \frac{\hbar^2}{2m} k_F^2 = \frac{\hbar^2}{2m} \left( \frac{3\pi^2 N}{V} \right)^{2/3} $$

(6.120)

So

$$ \gamma = \frac{\pi^2 N k_B^2}{2 \epsilon_F} = \frac{1}{\left( \frac{3\pi^2 N}{V} \right)^{2/3} \frac{\hbar^2}{m}} $$

(6.121)

If we measure $\gamma$ in an experiment, we can use it to determine the mass of an electron

$$ m^{-1} = \frac{\gamma}{\pi^2 N k_B^2} \left( \frac{3\pi^2 N}{V} \right)^{2/3} \frac{\hbar^2}{m} $$

(6.122)

In real metals, this mass is typically different from the real electron mass (by a factor of 0.3-2). To distinguish the mass determined using this formula from the true electron mass, we call this mass the thermal effective mass ($m_{th}$).

The reason that this mass is different from the real electron mass is because the electrons in a real metal are not free electrons. Instead, we have

- A lattice background
- Interactions between electrons
- Interactions between electrons and phonons

Although we ignored all these effects, the free electron theory discussed above turns out to be valid in many solids (most metals and semiconductors we have in our daily life). It turns out that indeed we can treat the electrons in a metal as a bunch of free fermions. These free fermionic particles are NOT electrons. They are called “quasi-particles”.

**Landau Fermi liquid theory**

- The electrons in a metal can be considered as a gas formed by non-interacting quasi-particles.
- The total number of these quasi-particles is identical to the total number of electrons. (Known as the Luttinger theorem).
- These quasi-particles may carry electric charge $\pm e$ and they have spin-1/2 (they are charged fermions).
- The mass of these quasi-particles (the effective mass) differs from the electron mass. The value of the effective mass depends on materials, electron density, temperature, etc.
- (Not going to be discussed in this course) The quasi-particles have a Fermi residue which is smaller than 1 but larger than 0.

Fermi residue is defined as

$$ Z = \lim_{\delta \to 0} \left[ f(\mu - \delta) - f(\mu + \delta) \right] $$

(6.123)

where $f(\epsilon)$ is the distribution function of the particles. For free fermions, $Z = 1$, because $f(\mu - \delta) = 1$ and $f(\mu + \delta) = 0$. But for interacting fermions, this value will be reduced by the interaction effect. However, for a Landau Fermi liquid, the value of $Z$ shall always be positive ($0 < Z < 1$). Quantum liquids with $Z = 0$ are NOT Landau Fermi liquids. They are one of the central focuses in current research (starting from the 80s).

This free fermion theory works surprisingly well in metals and semiconductors.

**Heavy fermion materials**

In typically metallic materials, the effective mass is not very different from the electron mass (differs by a factor or 0.3-2). But there are some materials, which are known as heavy-fermion compounds. In those materials, the effective mass is about 100 times larger than the electron mass.
Non-Fermi liquids

It is also worthwhile to notice that there are some other materials, in which these Landau Fermi liquid theory totally fails. This liquids are known as the non-Fermi liquids.

Landau Fermi liquid is very powerful, but it also enforce certain limitations to our materials. For example, as will be discussed in latter parts, some materials can turn into a superconductor (a conductor with zero resistivity) at low temperature $T < T_c$. For Landau’s Fermi liquids, the value of $T_c$ cannot exceeds $10 \sim 20$ K (about $\sim 260$ degree), which implies that we cannot use these superconductors on an industrial scale, because we cannot run our machines at such a low temperature. In the 80s, people realized that for non-Fermi liquids, such a constrain will no longer exist, so we can have superconductors at a much higher temperature. These materials are known as high temperature superconductors (high $T_c$). Currently, the highest $T_c$ is about 200 K (about $\sim 70$ degree).

Increase the transition temperature from a few K to 200 K is very important. The reason is because to cool down a system down to a few K, one need to use very expensive methods (e.g. liquid Helium). But to reach the temperature of 100-200K is extremely easy. One just need to dip the sample into liquid Nitrogen. Nitrogen is very easy to obtain (78% of our air is made by Nitrogen). The price for liquid Nitrogen is comparable with beer. The boiling point of Nitrogen is about 77K. Liquid Nitrogen can help us reach 77K without any difficulty at low cost. So raising $T_c$ above 77K is a very important revolution, which is achieved in the late 80s, using non-Fermi-liquid materials.

6.1.10. Thermal conductivity

In the last chapter, we learned that the thermal conductivity for a gas is:

$$K = \frac{1}{3} \frac{C}{V l}$$  \hspace{1cm} (6.124)

We used this formula to compute the thermal conductivity of phonons. Here, we use the same formula to compute the thermal conductivity of electrons.

The heat conductivity of electrons is

$$C_{el} = \frac{\pi^2}{2} N k_B T \frac{k_B T}{\epsilon_F}$$  \hspace{1cm} (6.125)

and we also known that the mean free path $l = v_F \tau$.

So

$$K_{el} = \frac{1}{3} \frac{C}{V l} = \frac{1}{3} \frac{\pi^2}{2} N k_B T \frac{k_B T}{\epsilon_F} v_F (v_F \tau) = \frac{\pi^2}{3} \frac{N k_B^2 T}{m \epsilon_F} \frac{1}{\tau}$$

$$= \frac{\pi^2 n k_B^2 T}{3 m \tau}$$  \hspace{1cm} (6.126)

In a clean metal $K_{electron} \gg K_{phonon}$. In dirty metals, $K_{electron}$ is small because the collision time $\tau$ is small. In dirty metals, $K_{electron} \approx K_{phonon}$.

In a metal, $K_{metal} = K_{electron} + K_{phonon} \gg K_{phonon}$. In an insulator, $K_{electron} = 0$, so $K_{insulator} = K_{phonon}$. So we have $K_{metal} \gg K_{insulator}$. This is why metal is cold to touch in the winter.

The Wiedemann–Franz law.

The ratio between the heat conductivity and the electric conductivity is

$$\frac{K}{\sigma} = \frac{\pi^2 n k_B^2 T}{3 m \tau} = \frac{\pi^2 k_B^2 T}{3 e^2}$$  \hspace{1cm} (6.127)

We can define the Lorenz number:

$$L = \frac{\pi^2 k_B^2}{3 e^2} = 2.44 \times 10^{-8} \text{ W} \Omega / K^2$$  \hspace{1cm} (6.128)

and the ratio between $K$ and $\sigma$ is just

$$\frac{K}{\sigma} = L T$$  \hspace{1cm} (6.129)

This relation is known as the Wiedemann–Franz law.

In real materials (metals), the value of $L$ is typically very close to $2.44 \times 10^{-8} \text{ W} \Omega / K^2$. 