Obsolescence of Durable Goods and Optimal Consumption*

Ennio Stacchetti
New York University

Dmitriy Stolyarov
University of Michigan

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Abstract

We study a model with a durable good subject to periodic innovation, and characterize the optimal purchasing policy. The key result is that consumers optimally synchronize new purchases with the innovation cycle. Hence, some agents react to wealth shocks by adjusting only the timing of durable purchases, while other agents adjust only their non-durable consumption. Consequently, elasticity of demand for durables is not monotonic in individual wealth. On aggregate, demand for durables starts responding to a wealth shock only after a lag, and this response is larger for goods that are more lumpy, longer-lived and have steeper depreciation profiles.

Our key synchronization result generalizes in a setting where innovations arrive randomly. On each arrival, an initial burst of durable demand is followed by an interval with no purchases. This purchasing behavior looks like a bandwagon effect, but here it arises from obsolescence rather than externality.

1 Introduction

Obsolescence is the major reason for depreciation in markets with technological innovation.1 Since much of this innovation is incorporated in new durables, modeling obsolescence of durable goods is vital for our understanding of macroeconomic effects.

How is obsolescence different from physical depreciation? One key distinction is in the depreciation pattern. The service flow from a durable is a function of the vintage of its technology as well as of the age of the good itself. Vintage plays a separate role in determining the service flow in the industries where technology changes periodically. One example of such an industry is computers: many old computers are considered useless, even though they are still as functional as when they were new. Another key feature of obsolescence is simultaneous depreciation of individual units. While physical depreciation is idiosyncratic and its aggregate effects are smooth, obsolescence affects all durables at the same time. For

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1Empirically, obsolescence manifests itself as a falling quality-adjusted price of a durable. For example, during 1960-2000, the annual obsolescence rate has been 23.5% for computers, 8.7% for communications equipment and 2.5% for automobiles (Cummins and Violante, 2002, Table II).
example, if one takes a mass of lightbulbs, the stock of useful bulbs will decay smoothly, even though each individual unit dies abruptly. By contrast, the whole stock of analog TVs will depreciate at the same date when the broadcasting switches to digital format (HDTV). We focus on the case where obsolescence has the most distinct aggregate effects - that of durables subject to periodic and abrupt obsolescence.

While in reality obsolescence patterns have both discrete and continuous elements, markets where discrete obsolescence is likely to be important are commonplace. These include at least four (overlapping) categories. (1) Markets that are fully or nearly monopolized, such as those for computer processors and software. When development of new products involves fixed costs, it is optimal for the monopolist to innovate only periodically (Fishman and Rob, 2000). (2) Markets where new products have a different format or standard. Format switching is typical for data recording or storage devices, such as disk drives, camcorders and digital cameras. (3) Goods that depend on a “bottleneck” (lagging) technology. For example, power supply has been a constraining factor in adding new features to many portable electronic devices. (4) Markets where technological constraints are imposed by government regulation, such as cellular communications. New generations of products can be introduced only when fresh radio spectrum becomes available.

What should happen to the demand for durables in a world where obsolescence can be discrete? If all individual units are expected to depreciate abruptly at some future date, consumers who purchase their durables just before this date will enjoy a lower service flow than those who buy soon after. Hence, consumers have an incentive to buy a durable only when the design is sufficiently new and is not about to be changed soon. Thus demand for new durables should be concentrated around the dates when the new models become available. The coordination of purchases with innovation dates is our key result that leads to unique predictions about demand fluctuations and propagation of shocks.

We consider an economy with infinitely lived agents who consume a durable and a nondurable good. Agents differ in their permanent income level, and can borrow and lend at an exogenously given interest rate. The durable is produced by a competitive industry with CRS technology. There are no secondary markets for used durables; units that are replaced are thrown away. Consequently, durables are purchased infrequently because the service from a current unit acts as a fixed opportunity cost of adjustment.

We solve analytically for the optimal consumption paths of individuals. Consumers optimally synchronize their new durable purchases with the design cycle. Although durables can be replaced at any time, consumers only purchase them at dates when new models are introduced. That is, agents only choose holding periods that are multiples of the design cycle length. Since the relevant choices of holding periods are discrete, the consumers smooth consumption by alternating between two holding periods from time to time.

Consumers endogenously partition themselves into classes according to their wealth and the age of their durable goods, with each class following a different durable replacement rule. Two types of rules are optimal. One type, which we term a “fixed” rule, is an \((s, S)\) policy with a constant replacement frequency. The other type is a “flexible” rule that alternates between two adjacent fixed rules at irregular intervals.

A key difference between the two types of rules is how the agents react to unexpected

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2This strong form of synchronization arises because the design cycle length is constant and certain and because the relative price of the durable does not change over time. Relaxing either one of these assumptions will sometimes, but not always, induce purchases in the middle of the design cycle. See Section 5 for details.
changes in wealth. Consumers that follow a fixed rule adjust only their non-durable consumption in response to a marginal windfall. By contrast, consumers that follow flexible rules adjust only their durable consumption.

In particular, if a fixed rule consumer receives a windfall, he will immediately change his non-durable consumption by the annuity value of this windfall. In contrast, a flexible rule consumer will save the windfall and spend it, after a delay, on purchasing some future durable one period earlier. Consequently, it appears that, on aggregate, total consumption adjusts insufficiently to innovations in wealth, and that durable consumption adjusts with a lag. In addition, discrete obsolescence makes adjustment lags longer, because durable purchases do not happen until the next new model arrival.

The magnitude of response of aggregate durable consumption to wealth shocks depends on the mass of consumers in fixed and flexible rule classes. We show that, on aggregate, durable demand is more wealth elastic for goods that are more lumpy, longer-lived and subject to faster obsolescence. Demand for such goods will exhibit a stronger response to a permanent price cut, for example.

In the model, fixed rule classes and flexible rule classes are interlaced. This gives rise to another interesting implication: elasticity of demand for the durable is non-monotonic in wealth. In particular, the richest consumers have zero elasticity of demand, because they already purchase every new model available. However, consumers who use other fixed rules exhibit zero elasticity as well, including the poorest group who do not purchase the durable at all. This means that offering a price cut to the poorest consumers in order to boost sales is only marginally effective; the producers should instead target “frequency switchers” that use flexible rules.

We modify the basic model to allow for stochastic innovation and show that there is a fundamental reason for synchronized purchases. The synchronization arises when the expected service flow from the durable depends on when the durable is purchased. If, for example, the hazard rate of a new model arrival is increasing over time, consumers know that by delaying a purchase of the current model they increase the chance of getting a lower service flow from it. In this case, consumers have an incentive to purchase the durable earlier. We think that many innovation processes have hazard rates that are negligible immediately after an innovation; after all, no one expects a new generation of products to appear immediately after the introduction of a new model.

We therefore assume that there is a minimum gestation period when no innovations can happen, and after that new models arrive via a Poisson process. We show that when a new model arrives, consumers choose to buy it either with a very short delay or a very long delay. Therefore, on arrival, there is an initial burst in demand, followed by an interval where no purchases are made. This behavior matches the intuition from our basic model. Consumers cease to purchase durables when the date of possible new model arrival draws sufficiently close, and would rather postpone their purchase until the next arrival. We show that the no-purchase interval is always longer than the minimum gestation period. Of course, if the period between arrivals turns out to be unexpectedly long, consumers will eventually accumulate a lot of wealth and purchase the existing model. This is why purchases restart after a long delay if no innovation has occurred.

In many markets, obsolescence occurs not just because innovations make goods better, but also because the existing durables become cheaper to produce. We therefore allow the price of the new good to fall over time. Falling price gives consumers an incentive to postpone
purchases, and if it falls sufficiently fast, some agents choose to buy the new model with a delay. Nevertheless, our numerical results show that for a range of plausible parameter values the no-purchase interval is longer than the average design cycle length. This means that for most realizations of the innovation times, the time series of demand for durables will look similar to the basic model: short bursts at innovation dates, followed by periods with no purchases.

Our model shows how the aggregate bunching of purchases can arise without a network externality. In the model, synchronized behavior occurs because depreciation episodes are correlated across individual goods. Therefore, we can expect to observe this behavior in a different (and, perhaps, larger) set of markets than predicted by the externality story.

Our work is related to a large literature that studies models with infrequent replacement of durable goods. Most of this literature considers optimal \((s, S)\) replacement policies. There are three broad categories of related \((s, S)\) models. The first category includes representative agent models with a budget constraint (e.g. Grossman and Laroque (1990), Eberly (1994)). These models have only one good, the durable, and thus look at durable consumption separately. The second category includes replacement models with aggregate dynamics (e.g. Caballero and Engel (1999), Caplin and Leahy (1999), Adda and Cooper (2000)). These papers consider a replacement problem without an inter-period budget constraint. The model of Adda and Cooper (2000) includes durables and non-durables, but does not allow borrowing and lending. The third body of literature (e.g. Caballero (1993), Attanasio (2000)) does not consider the optimal replacement problem but assumes that the optimal replacement policy for the durable is an \((s, S)\) rule. In addition to introducing a new model, we develop a solution methodology that can be used in other replacement problems with indivisibilities. For example, an investment problem with discrete obsolescence is a special case of our model.

Our work also contributes to a broader macroeconomic literature that studies the interaction of durable and non-durable consumption and the propagation of income and wealth shocks. In our model, periodic obsolescence determines the optimal timing of durable purchases, and this, in turn, affects how shocks propagate. Leahy and Zeira (2000) derive a closely related result in a framework where consumers buy the durable only once in their lifetime. They find what they call an “insulation effect”: both non-durable consumption and the size of the durable are unaffected by wealth shocks, but the timing of purchases is. In our model, non-durable consumption is insulated from wealth shocks for individuals who follow flexible rules. We think that insulation effects are likely to arise when consumers face constraints on how they can adjust to changes in wealth. In Leahy and Zeira’s paper the constraint is that consumers cannot purchase the durable repeatedly, and in our model consumers cannot choose the purchase size. The constraints seem to make some margins of adjustment more important than others. Consequently, consumers may ignore some adjustment margins that are not constrained. The reaction to a shock is then “channeled” to compensate for the presence of a constraint.

Section 2 describes the model. Section 3 separately solves the durable consumption problem. We construct optimal policies using a very simple geometric argument. Section 4 determines the optimal allocation of wealth between durable and non-durable consumption. Section 5 discusses our results. In Section 6 we analyze the model with stochastic innovation and discuss its implications. Section 7 concludes.
2 Model

We consider a dynamic economy with two goods, a durable and a non-durable good, and a continuum of agents that differ in their permanent income \( y \in [y, \bar{y}] \). Incomes are given exogenously, and they stay constant over time.

**Goods, technology and preferences:** The durable good is indivisible and is produced by a constant returns to scale technology that uses \( p_0 \) units of the non-durable good for each unit of the durable good. New durables (new models) are introduced regularly into the market at times \( \tau \in \mathbb{N} = \{0, 1, \ldots\} \). Without loss of generality, we have normalized to 1 the length of a design cycle. We refer to the durable introduced at time \( \tau \) as “model \( \tau \”). The technological age of a durable good is the number of new models introduced since it was produced. The consumers are infinitely-lived and have a (common) discount rate \( \rho \) and a (common) separable flow utility function \( v(\alpha, c) = x_\alpha + u(c) \), where \( \alpha \in \{0, 1, \ldots, T\} \) denotes the technological age of the durable good, and \( c \) is the consumption flow for the non-durable. Durable goods of any age less than \( T \) are perfect substitutes and each agent consumes at most one unit (additional units provide no utility). We think of the non-durable good as money for the consumption of other goods, and of \( u \) as an indirect utility function. We assume that \( u' > 0, u'' < 0, u'(0) = \infty \), and \( x_0 \geq x_1 \geq \cdots \geq x_{T-1} > x_T = 0 \).

Obsolescence is the only form of depreciation in our model. A durable becomes useless when its technological age is \( T \) or more. A new model \( \tau \) provides a flow service of \( x_0 \) in the period \( [\tau, \tau + 1) \). When a new model is introduced at time \( \tau + 1 \), model \( \tau \)'s flow service decreases to \( x_1 \), and so on. The consumers can buy a new durable at any moment, but the durable is depreciated as soon as the new model is introduced. Thus, if a consumer buys a new durable at time \( t \in [\tau, \tau + 1) \), he gets the flow service \( x_0 \) in the interval \( [t, \tau + 1) \), and then the flow service \( x_1 \) in the interval \( [\tau + 1, \tau + 2) \), and so on, as long as he doesn’t replace the durable.

If \( x \) is interpreted as service flow, then one has to assume that the durable becomes less useful when new models arrive. However, our model also admits the interpretation of \( x \) as *relative utility* of a durable with respect to the latest model. In Appendix 1 we show that if the flow service of a durable remains constant over its lifetime, we can re-normalize utility and define \( x_\alpha \) as the relative service flow from a durable of age \( \alpha \). Because utility is assumed to be quasilinear, this normalization changes only the utility level, but not the consumer’s ranking of different durable consumption trajectories. In the example of Appendix 1, \( x_\alpha = g(T - \alpha) \), where \( g \) is the average rate of technical progress in durables. In this case, each subsequent model of the durable has \( e^g \) times higher service flow than the previous one, and \( x \) falls with \( \alpha \) simply because better goods become available at the same price.

The consumers can borrow and lend, but there are no secondary markets for used durables.

**Prices:** Since the production technology is CRS, the price ratio of the durable good to the non-durable good is equal to a constant \( p_0 \) at all times. We will assume that the interest rate is fixed and equal to the discount rate: \( r(t) = \rho \) for all \( t \geq 0 \). We therefore perform a partial equilibrium analysis. We think of the market for durables as being a small part of the aggregate economy and hence ignore the effect of durable demand on the interest rate. Our choice of interest rate is consistent with stationary equilibria. In a general equilibrium model where income (resource) flow and production technology are constant over time, a
stationary equilibrium would imply a constant interest rate equal to the discount rate. If \( q(t) \) and \( p(t) \) denote, respectively, the prices of the non-durable and durable goods at time \( t \), our assumption of a constant interest rate implies that \( q(t) = e^{-\rho t} \) and \( p(t) = p_0 q(t) \) for all \( t \geq 0 \), where we have normalized so that \( q(0) = 1 \). Define the total discount rate for one period as \( \beta = e^{-\rho} \).

**Consumer problem:** Given his initial state \((\alpha, w)\), where \( \alpha \in \{1, \ldots, T\} \) is the age of his endowed durable and \( w \) is his total wealth, a consumer chooses a sequence of durable purchase dates and a non-durable consumption path to maximize his discounted lifetime utility, \( \int_0^\infty e^{-\rho t}[x_{\alpha t} + u(c_t)]dt \), subject to a lifetime budget constraint. An agent’s current wealth is equal to the present discounted value of all his future earnings, \( y/\rho \), minus the present discounted value of his debts (past borrowing minus lending).

Since \( r(t) = \rho \) for all \( t \) and utility is additively separable, optimally, non-durable consumption must be constant over time. Indeed, the (necessary and sufficient) first-order condition for non-durable consumption is in this case \( e^{-\rho t}u'(c(t)) = \lambda e^{-\rho t} \) for all \( t \), where \( \lambda > 0 \) is the Lagrange multiplier on the budget constraint. This implies that \( c(t) = c(0) \) for all \( t > 0 \).

Let \( \hat{u}(c) \) be the discounted non-durable consumption utility over one period (of length 1) in which a consumer spends (optimally) a budget \( c \). This budget affords the constant consumption flow \( cp/(1-\beta) \). Hence

\[
\hat{u}(c) = \int_0^1 e^{-\rho t}u \left( \frac{\rho c}{1-\beta} \right) dt = \left[ \frac{1-\beta}{\rho} \right] u \left( \frac{\rho c}{1-\beta} \right).
\]

Let the consumer spend a constant non-durable budget \( c \) per period. Then, his lifetime non-durable discounted utility and total budget are respectively \( \hat{u}(c)/(1-\beta) \) and \( c/(1-\beta) \), and his residual budget for the consumption of durables is \( b = w - c/(1-\beta) \).

Let \( V_\alpha(b) \) denote the optimal durable consumption utility of a consumer that is endowed with a good of age \( \alpha \) and spends a total budget \( b \) on durables. Then the problem of an agent with initial state \((\alpha, w)\) is

\[
U_\alpha(w) = \max_{c \in [0, w]} \frac{\hat{u}(c)}{1-\beta} + V_\alpha \left( w - \frac{c}{1-\beta} \right).
\] (1)

In Section 3, we explicitly construct the functions \( V_\alpha, \alpha \in \{1, \ldots, T\} \), and in Section 4 we obtain the full solution for problem (1).

### 3 Durable consumption problem

**Discrete Time:** As a preliminary step in analyzing the durable consumption problem, we study a discrete time problem where the consumers are arbitrarily constrained to make new purchases only at the beginning of every period, that is, at times \( t \in \mathbb{N} \). We subsequently show that removing this restriction does not change the optimal durable purchasing policy.

A consumer must choose the periods when he purchases a (new) unit of the durable good. A durable purchasing policy \( \delta = \{\delta_t\}_{t \geq 0} \) specifies the periods in which the agent buys a new unit (\( \delta_t = 1 \)) or keeps the old unit he has (\( \delta_t = 0 \)). For any \( i, j \in \mathbb{N} \), let \( i \oplus j = \min\{i + j, T\} \) and \( i \ominus j = \max\{i - j, 0\} \). Given an initial unit of age \( \alpha \), a purchasing policy determines the
age of the unit consumed in every period \( t \geq 0 \) recursively as follows: \( \alpha_t = 0 \) if \( \delta_t = 1 \) and \( \alpha_t = \alpha_{t-1} + 1 \) if \( \delta_t = 0. \)

The optimization problem of an agent that initially has a good of age \( \alpha \) and durable budget \( b \) is

\[
V_\alpha(b) = \max \sum_{t \geq 0} \beta^t x_{\alpha_t}
\]

s.t. \( \alpha_{-1} = \alpha - 1 \), \( \delta_t \in \{0, 1\} \) and \( \alpha_t = (1 - \delta_t)[\alpha_{t-1} + 1] \), \( t \geq 0 \)

\[
b = p_0 \sum_{t \geq 0} \beta^t \delta_t,
\]

where \( x_\alpha = x_\alpha(1 - \beta)/\rho \) denotes the total discounted utility from the consumption of a durable of age \( \alpha \) over one period.

We solve the potentially difficult integer programming problem above using a direct geometric argument focusing on a particularly simple class of policies.

**Definition:** For each \( R = 1, \ldots, T \), a policy \( \delta \) that replaces the durable every time it reaches age \( R \) is called an \( R \)-fixed rule. That is, \( \delta \) is an \( R \)-fixed rule if for all \( t \), \( \delta_t = 1 \) if and only if \( \alpha_{t-1} = R - 1 \). A \((T + 1)\)-fixed rule is to never replace the durable: \( \delta_t = 0 \) for all \( t \).

Let \( X_{\alpha,R} \) denote the total discounted utility from holding a durable from age \( \alpha \) until age \( R \):

\[
X_{\alpha,R} = \begin{cases} 
\sum_{t=\alpha}^{R-1} \beta^{t-\alpha} x_t & \text{if } \alpha < R \\
0 & \text{if } \alpha \geq R.
\end{cases}
\]

For \( R \leq T \), the value of following the \( R \)-fixed rule starting with a useless durable \((\alpha = T)\) equals \( v_{T,R} = X_{0,R}/(1 - \beta^R) \), and its corresponding budget is \( b_{T,R} = p_0/(1 - \beta^R) \). The value and budget of the \((T + 1)\)-fixed rule are both zero.

Construct a piecewise linear function by joining the adjacent points \((b_{T,R+1}, v_{T,R+1})\) and \((b_{T,R}, v_{T,R})\) \((1 \leq R \leq T)\) with straight lines. Theorem 1 below states that this piecewise linear function is \( V_T \). Moreover, \( V_T \) is concave (see the left frame of Figure 1).

Assume that \( \alpha = T \) and for an arbitrary purchasing policy \( \delta \), group purchases by their “replacement age”. That is, for each \( R = 1, \ldots, T \), let \( L_R \) be the purchase dates of all durables that are later replaced at age \( R \). Compute the weight \( \lambda_R = (1 - \beta^R) \sum_{t \in L_R} \beta^t \) and let \( \lambda_{T+1} = 1 - \sum_{R=1}^{T} \lambda_R \). Roughly, the weight \( \lambda_R \) corresponds to the fraction of purchases that result in the replacement of a durable at age \( R \). For example, if the policy is an \( R \)-fixed rule with \( R < T + 1 \), then \( L_R \) contains all the periods \( t \) where \( \delta_t = 1 \), so that \( \lambda_R = 1 \) and \( \lambda_k = 0 \) for all \( k \neq R \). Let \((b, v)\) denote the budget and value of policy \( \delta \). It turns out that:

\[
\begin{bmatrix} b \\ v \end{bmatrix} = \sum_{R=1}^{T+1} \lambda_R \begin{bmatrix} b_{T,R} \\ v_{T,R} \end{bmatrix}.
\]

Since the weights \( \lambda_R \) are nonnegative and add up to 1, the right-hand side is a convex combination of the points \( \{(v_{T,R}, b_{T,R})\}_{R=1}^{T+1} \). That is, the point \((b, v)\) must be in the convex hull of \( \{(v_{T,R}, b_{T,R})\}_{R=1}^{T+1} \), as depicted in Figure 1. Note that the upper frontier of this set.

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3To deal with period 0 as with any other period, we specify the age that the endowed durable would have been in the “previous period” and allow \( \alpha_{-1} = -1 \) to include the case \( \alpha = 0 \).
δ replaces durables only when they are of age \( R \) or \( R + 1 \) only. Such a policy is called an \( R \)-flexible rule. Its corresponding weights satisfy \( \lambda_k = 0 \) for all \( k \not\in \{ R, R + 1 \} \). By appropriately choosing the periods when durables of age \( R \) or age \( R + 1 \) are replaced, we can also ensure that 

\[
\delta^* = \lambda_R^* b_{T,R} + \lambda_{R+1}^* b_{T,R+1}
\]

(as we explain later, this is always possible provided that \( \beta \) is sufficiently large). Then, the value of \( \delta^* \) is 

\[
\lambda_R^* v_{T,R} + \lambda_{R+1}^* v_{T,R+1} = V_T(b). \]

That is, \( \delta^* \) is optimal for the budget \( b \).

For an arbitrary \( \alpha \) now, let \( b_{a,R} \) and \( v_{a,R} \) denote the cost and the value of following the \( R \)-fixed rule when the endowed durable is of age \( \alpha \). Then

\[
\begin{bmatrix}
    b_{\alpha,R} \\
    v_{\alpha,R}
\end{bmatrix} = \begin{bmatrix}
    0 \\
    X_{\alpha,R}
\end{bmatrix} + \frac{\beta^{R+\alpha}}{1 - \beta^R} \begin{bmatrix}
    p_0 \\
    X_{0,R}
\end{bmatrix}
\]

for all \( R \leq T \)

and \( (b_{\alpha,T+1}, v_{\alpha,T+1}) = (0, X_{\alpha,T}) \). It is also convenient to define \( b_{T+1,R+1} = p_0 \) and \( b_{0,1} = \beta p_0 / (1 - \beta) \). Rules that replace goods more frequently require bigger budgets and have higher values. Hence \( b_{a,R} > b_{a,R+1} \) and \( v_{a,R} > v_{a,R+1} \).

The piecewise linear function obtained by joining the adjacent points \( (b_{\alpha,R+1}, v_{\alpha,R+1}) \) and \( (b_{\alpha,R}, v_{\alpha,R}) \) \( (1 \leq R \leq T) \) with straight lines is the optimal value function \( V_\alpha \) (see Theorem 1 below).

Figure 1 (right frame) presents simultaneously the optimal value functions \( V_1, V_2 \) and \( V_3 \) for the case when \( T = 3 \).

**Definition:** Let \( 1 \leq R \leq T - 1 \) and \( b \geq 0 \). A policy \( \delta \) is an \((R,b)\)-flexible rule if it replaces durables only when they are of age \( R \) or age \( R + 1 \) and spends the budget \( b \) exactly. If \( \delta \) is an \((R,b)\)-flexible rule then for all \( t \), \( \delta_t = 1 \) implies that \( \alpha_{t-1} \in \{ R - 1, R \} \).

Since an \((R,b)\)-flexible rule sometimes replaces goods at age \( R \), and sometimes at age \( R + 1 \), it costs more than an \((R + 1)\)-fixed rule but less than an \( R \)-fixed rule. Hence, when the endowed good is of age \( \alpha \), \( b \) must be in the interval \([b_{\alpha,R+1}, b_{\alpha,R}]\). As we will see, for a certain range of \( b \) within this interval, there are multiple \((R,b)\)-flexible rules. The \( R \)-fixed and the \((R + 1)\)-fixed rules are both special cases of the \((R,b)\)-flexible rule for \( b = b_{T,R} \) and \( b = b_{T,R+1} \), respectively.
For $1 \leq \alpha, R \leq T$, let
\[
A_R = \frac{v_{\alpha,R} - v_{\alpha,R+1}}{b_{\alpha,R} - b_{\alpha,R+1}} = \frac{1}{\rho_0} \left[ X_{0,R} - \hat{x}_R \left[ \frac{1 - \beta^R}{1 - \beta} \right] \right].
\]
Note that $A_R$ is independent of $\alpha$ and equals the slope of $V_\alpha$ on $[b_{\alpha,R+1}, b_{\alpha,R}]$. It is easy to check that $A_T > A_{T-1} > \cdots > A_1 > 0$, and therefore $V_\alpha$ is indeed a concave function.

**Theorem 1:** Assume that
\[
\beta^{T-1}(1 + \beta) > 1.
\]
For each $\alpha = 1, \ldots, T$, the optimal value function $V_\alpha$ is
\[
V_\alpha(b) = v_{\alpha,R+1} + A_R(b - b_{\alpha,R+1}), \quad b \in [b_{\alpha,R+1}, b_{\alpha,R}], \quad R = T, \ldots, 1,
\]
and for any budget $b \geq 0$, a corresponding optimal purchasing policy is an $(R, b)$-flexible rule, where $R$ is such that $b \in [b_{\alpha,R+1}, b_{\alpha,R}]$ (when $b = b_{\alpha,R}$, this policy coincides with the $R$-fixed rule). More precisely, the optimal purchasing policy is given by
\[
\delta^*_\alpha(b) = \begin{cases} 
0 & \text{for } b < b_{\alpha+1,\alpha+1} \\
\{0, 1\} & \text{for } b_{\alpha+1,\alpha+1} \leq b \leq b_{\alpha-1,\alpha} \\
1 & \text{for } b > b_{\alpha-1,\alpha}.
\end{cases}
\]

**Proof:** See Appendix 1.

Assumption (2) is equivalent to $\beta > \bar{\beta}$, where $\bar{\beta}$ is the (unique) root of $\beta^{T-1}(1 + \beta) = 1$. This is the same as assuming that $\rho < \bar{\rho}$, where $\bar{\rho} = e^{-\bar{\beta}}$. When $\beta$ is relatively small, there are budgets $b$ that do not correspond to any durable purchasing policy. The intuition is clear. Suppose $\beta$ is close to 0. Then the durable budget is almost fully determined by the timing of the first purchase. Let $\alpha < R$, $\delta$ be an $R$-flexible rule, and $b$ be its corresponding budget. If the first purchase happens when the good is of age $R$, then $b \sim b_{\alpha,R}$ (even if all subsequent purchases replace durables of age $R + 1$), and if it happens at age $R + 1$, then $b \sim b_{\alpha,R+1}$ (even if all subsequent purchases replace durables of age $R$). Hence, budgets around the middle of the interval $(b_{\alpha,R+1}, b_{\alpha,R})$ are unattainable.

An agent that follows an $R$-flexible rule replaces goods of age $R$ or $R + 1$, but he is not always indifferent between these replacement ages. To follow an $R$-flexible rule requires that in each period the agent maintain a budget that is compatible with this rule. Assume that the durable has reached age $R$ in the current period. Then, the current budget $b$ must be in the interval $[b_{R,R+1}, b_{R,R}]$. Suppose $b$ is close to $b_{R,R}$. If the agent keeps the good this period, his budget next period would be $b/\beta > b_{R+1,R}$, too large to follow the $R$-flexible rule from that point onward. Therefore, the agent can keep the durable this period only if $b \in [b_{R,R+1}, b_{R-1,R}]$; if $b > b_{R-1,R}$, the agent must replace now at age $R$. Now assume that $b$ is close to $b_{R,R+1}$. If the agent replaces the durable now, his budget next period would be $(b - p_0)/\beta < b_{R+1,R}$, too small to follow the $R$-flexible rule from that point onward. Therefore, the agent can replace his durable of age $R$ this period only if $b \in [b_{R+1,R+1}, b_{R,R}]$; if $b < b_{R+1,R+1}$, the agent must keep the durable for one more period. Assumption (2) also guarantees that $b_{R+1,R+1} < b_{R-1,R}$, and for $b \in [b_{R+1,R+1}, b_{R-1,R}]$ both keeping and replacing the durable this period are consistent with the $R$-flexible rule. For this interval of budgets, the agent is indifferent between replacing the durable now at age $R$ and next period at age $R + 1$. 

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Continuous Time: We now allow consumers to purchase durables at times other than \( t \in \mathbb{N} \) and show that this does not change the optimal value function. For the continuous time replacement problem, we need a more detailed representation of the durable purchasing policy. Let \( \tau_k^d \) denote the period (or, equivalently, the model number) and \( d_k \in [0, 1) \) be the “delay” of the \( k \)-th purchase, so the time of the \( k \)-th purchase is \( \tau_k^d + d_k \). The following theorem states that it is optimal to set \( d_k = 0 \) for all \( k \).

**Theorem 2:** For each \( \alpha = 1, \ldots, T \), the optimal value function is

\[
V_\alpha(b) = v_{\alpha,R+1} + A_R(b - b_{\alpha,R+1}), \quad b \in [b_{\alpha,R+1}, b_{\alpha,R}], \quad R = T, \ldots, 1.
\]

For any budget \( b \geq 0 \), the corresponding optimal purchasing policy \( \{ (\tau_k^d, d_k) \}_{k \geq 1} \) has \( d_k = 0 \) for all \( k \) and is an \(( R, b )\)-flexible rule, where \( R \) is such that \( b \in [b_{\alpha,R+1}, b_{\alpha,R}] \).

**Proof:** See Appendix 1.

The idea of the proof is as follows. When the consumer decides whether to delay by \( d \) a durable purchase, he weights the loss of service flow against the financial gain of paying for the durable later. When \( r = \rho \), the financial gain is less than the corresponding loss of service (in fact, the result holds as long as the interest rate is not too high relative to \( \rho \)). An arbitrary policy with delays can be modified recursively by eliminating one delay at a time while maintaining the same budget and improving its value.

### 4 Optimal budget allocation

We now solve problem (1) for the optimal consumption of non-durables as a function of \( \alpha \) and \( w \). An agent with wealth \( w \) that spends \( b \) on durables optimally spends \( c = (1 - \beta)(w - b) \) per period on non-durables. Ideally, the agent should pick \( c \) (or, equivalently, \( b \)) so as to equate the marginal utility of consumption \( \hat{u}'(c) \) and the marginal utility of wealth \( V''_\alpha(b) \). Figure 2 depicts the marginal utility of wealth (the falling step-function because \( V_\alpha \) is a concave piecewise linear function) and the marginal utility of consumption as functions of \( b \) (for given values of \( \alpha \) and \( w \)). In the figure, \( \hat{u}' \) crosses \( V''_\alpha \) at a point of discontinuity. This depicts the situation when the optimal durable budget equals \( b_{\alpha,R} \) and the corresponding durable purchasing policy is the \( R \)-fixed rule. Now decrease \( w \) by a small amount. The graph of \( \hat{u}'((1 - \beta)(w - b)) \) will shift to the left, but it will still cross \( V''_\alpha \) at \( b = b_{\alpha,R} \). In other words, there is an interval of wealths \( w \) for which it is optimal to follow the \( R \)-fixed rule in the state \(( \alpha, w )\). If we further decrease \( w \), \( \hat{u}' \) will eventually cross \( V''_\alpha \) at a point where \( V''_\alpha \) is flat and equal to \( A_R \). This is the case when it is optimal to choose a budget corresponding to an \( R \)-flexible rule and pick the non-durable budget \( c_R \), where \( \hat{u}'(c_R) = A_R \). Hence, there is also an interval of wealths \( w \) for which it is optimal to follow the \( R \)-flexible rule and spend \( c_R \) in non-durables every period. For that range of wealths, the optimal non-durable budget remains constant and variations of wealth affect the durable consumption path only (higher wealths afford replacing durables at age \( R \) more frequently, while lower wealths require replacing durables at age \( R + 1 \) more often). In contrast, when a fixed rule is optimal, a higher wealth leads to a higher level of non-durable consumption.

For a fixed \( \alpha \), if \( w \) varies continuously from infinity to zero, the intersection of \( \hat{u}' \) with \( V''_\alpha \) in Figure 2 moves monotonically to the left and maps out the optimal durable replacement rule (as a function of \( w \)). The wealthiest consumers use a \( 1 \)-fixed rule. Next comes a group
of consumers that follow 1-flexible rule, and then a group that follows the 2-fixed rule, and so on. The intervals of wealth where agents follow fixed rules are interlaced with the intervals of wealth where they follow flexible rules. The bounds of these intervals can be computed explicitly. Fix $\alpha$ and let

$$w_{\alpha,R}(c) = \frac{c}{1-\beta} + b_{\alpha,R}$$

be the wealth required to follow the $R$-fixed rule and spend a constant non-durable budget $c$ per period when the initial durable is of age $\alpha$. The wealthiest person that follows the $R$-flexible rule replaces his durable every $R$ periods and consumes $c_R$. Hence his wealth is $w_{\alpha,R}(c_R)$. The poorest person that follows the $(R-1)$-flexible rule also replaces his durable every $R$ periods but consumes $c_{R-1} > c_R$, so that his wealth is $w_{\alpha,R}(c_{R-1}) > w_{\alpha,R}(c_R)$. In between, there are consumers with wealth $w \in [w_{\alpha,R}(c_R), w_{\alpha,R}(c_{R-1})]$ that follow the $R$-fixed rule. Each one spends the same durable budget $b_{\alpha,R}$ and the non-durable budget per period $c_{\alpha,R}(w) = (1-\beta)(w - b_{\alpha,R})$.

A consumer with more wealth than $w_{1,1}(c_1) = (c_1 + p_0)/(1-\beta)$ will replace his durable every period and spend more than $c_1$ per period in non-durables. We will assume that $\bar{y}/\rho \geq w_{1,1}(c_1)$, and define $\bar{w} = \bar{y}/\rho$ and $c_0 = (1-\beta)\bar{w} - p_0$. Similarly, a consumer with less wealth than $c_T/(1-\beta)$ will spend all his wealth in non-durable consumption. We will assume that $\bar{y}/\rho \leq c_T/(1-\beta)$, and define $\bar{w} = \bar{y}/\rho$ and $c_{T+1} = (1-\beta)\bar{w}$.

We can also express the optimal purchasing policy (3), stated in Theorem 1, as a function of wealth (and with abuse of notation denote this function by the same symbol $\delta^*_\alpha$). The following theorem states these results formally.

**Theorem 3:** Let $c_0 = (1-\beta)\bar{w} - p_0$, $c_{T+1} = (1-\beta)\bar{w}$, and for each $R = 1,\ldots,T$, let $c_R$ be such that $u'(c_R) = A_R$. Denote by $c^*_\alpha(w)$ the optimal solution of problem (1). Then, for $\alpha = 1,\ldots,T$,

$$c^*_\alpha(w) = \begin{cases} c_{\alpha,R}(w) & \text{for } w \in [w_{\alpha,R}(c_R), w_{\alpha,R}(c_{R-1})], \ R = T + 1,\ldots,1, \\ c_R & \text{for } w \in [w_{\alpha,R+1}(c_R), w_{\alpha,R}(c_R)], \ R = T,\ldots,1, \end{cases}$$


and
\[
\delta^*_\alpha(w) = \begin{cases} 
0 & \text{for } w < w_{\alpha+1,\alpha+1}(c_\alpha) \\
\{0, 1\} & \text{for } w_{\alpha+1,\alpha+1}(c_\alpha) \leq w \leq w_{\alpha-1,\alpha}(c_\alpha) \\
1 & \text{for } w > w_{\alpha-1,\alpha}(c_\alpha).
\end{cases}
\] (4)

**Proof:** See Appendix 1.

Over time, a consumer that follows an \(R\)-fixed rule has a constant holding time \(R\) and revisits the same points in the state space \((\alpha, w)\) every \(R\) periods. His wealth trajectory is cyclical. While the consumer keeps the current good, both \(\alpha\) and \(w\) increase, as the consumer “saves” for the next purchase. When the new durable is purchased, both \(\alpha\) and \(w\) go down, and the holding cycle starts again.

The time path for wealth of a consumer that follows an \(R\)-flexible rule is more erratic. Usually, his wealth trajectory is not cyclical: each time the durable reaches age \(R\), his wealth will be \(w_R = [w_0 - p_0] \beta^R < w_0\). If \(w_R > w_{R-1,R}\), he will have to replace the durable again. But eventually, if he continues to replace each time the durable reaches age \(R\), he will reach a state \((R, w)\), where \(w < w_{R+1,R+1}(c_R)\). At this point, he is forced to wait one more period. Thus, the agent will switch replacement frequencies erratically, as each time that his state is of the form \((R, w)\), his wealth level \(w\) is at a different place of the interval \([w_{R+1,R}(c_R), w_{R,R}(c_R)]\).

### 4.1 Consumption classes

The optimal policies partition the state space \((\alpha, w)\) into disjoint classes, with each class corresponding to a different durable replacement rule. All individuals in a class follow the
same rule and the trajectories of their states stay forever in the same class. For every $R \in \{1, \ldots, T + 1\}$ and $\alpha \in \{1, \ldots, R\}$, let

$$W_R^\alpha = [w_{\alpha,R}(c_R), w_{\alpha,R}(c_R-1)]$$

be the wealth levels of consumers that follow an $R$-fixed rule and currently have a durable of age $\alpha$. Similarly, for every $R \in \{1, \ldots, T\}$ and $\alpha \in \{1, \ldots, \min\{R+1, T\}\}$ let

$$W_{R,R+1}^\alpha = (w_{\alpha,R+1}(c_R), w_{\alpha,R}(c_R))$$

be the wealth levels of consumers that follow an $R$-flexible rule and currently have a durable of age $\alpha$. Note that for each $\alpha$, $\{W_R^\alpha\}_{R=1}^{T+1} \cup \{W_{R,R+1}^\alpha\}_{R=1}^{T}$ forms a partition of $[\underline{w}, \bar{w}]$. At the beginning of every period, agents with a state in $C_R = \bigcup_{\alpha=1}^{R} \{\alpha\} \times W_R^\alpha$ follow the $R$-fixed rule, and with a state in $C_{R,R+1} = \bigcup_{\alpha=1}^{R+1} \{\alpha\} \times W_{R,R+1}^\alpha$ follow the $R$-flexible rule. Note that after the initial period, nobody visits the states $\{\alpha\} \times W_R^\alpha$, $\alpha > R$, or the states $\{\alpha\} \times W_{R,R+1}^\alpha$, $\alpha > R + 1$. A consumer with one of these initial states has been endowed with a durable that is “too old” for his initial wealth level. The classes $C_R$ and $C_{R,R+1}$ are closed: if an agent follows the $R$-fixed rule ($R$-flexible rule) and his initial state is in $C_R$ (in $C_{R,R+1}$), then his state remains in $C_R$ (in $C_{R,R+1}$) forever. Figure 3 illustrates consumption classes for the case $T = 3$ and one of the corresponding optimal consumption function $c^* (w)$ described in Theorem 3. Three horizontal lines on the lower panel of figure 3 represent the state space $\{1, 2, 3\} \times [\underline{w}, \bar{w}]$. Bold lines indicate wealth intervals that belong to fixed rule classes, and thin lines indicate flexible rule classes. Class boundaries are marked by dashed lines. Dotted lines indicate the intervals in the state space that are empty in the long run.

## 5 Results

### 5.1 Consumption response to a change in wealth

Aggregate durable and non-durable consumption both respond to aggregate changes in wealth. Consumers in class $C_R$ have a fixed durable budget and a positive marginal propensity to consume non-durables (see Figure 3). Therefore, if any such consumer receives windfall income, he will spend it all on non-durable consumption. By contrast, consumers in a class $C_{R,R+1}$ have a zero marginal propensity to consume non-durables and a variable durable budget. The magnitude of the overall response of durable consumption to a change in wealth will depend on the mass of consumers in fixed and flexible rule classes. These masses, of course, are functions of the wealth distribution. To isolate the effect of the model’s parameters on the sensitivity of durable consumption, we assume a uniform distribution over the set of recurrent states (i.e. the states marked by solid lines on Figure 3). Then the mass of consumers in classes $C_R$ and $C_{R,R+1}$ $(R = 1, \ldots, T)$ are respectively

$$\mu(C_R) = \sum_{\alpha=1}^{R} [w_{\alpha,R}(c_R-1) - w_{\alpha,R}(c_R)]$$

$$\mu(C_{R,R+1}) = \sum_{\alpha=1}^{\min\{R+1, T\}} [w_{\alpha,R}(c_R) - w_{\alpha,R+1}(c_R)].$$
Also, define $\mu(C_{T+1}) = w_{T,T+1}(c_T) - w$. Then, the fraction of consumers that follow flexible rules is

$$\theta = \frac{\sum_{R=1}^{T+1} \mu(C_{R,R+1})}{\sum_{R=1}^{T+1} \mu(C_R) + \sum_{R=1}^{T} \mu(C_{R,R+1})}.$$ 

Given a small change in wealth, approximately $\theta$ consumers will adjust only their durable consumption and $1-\theta$ consumers will adjust only their non-durable consumption. The larger is $\theta$, the more sensitive is durable consumption to changes in wealth.

Assume that $x_\alpha = g(T-\alpha)$, $\alpha = 0, \ldots, T$. This is the linear obsolescence pattern that arises when $x_\alpha$ is interpreted as a relative service flow (see Appendix 1). In this case, $g$ represents the obsolescence rate – the speed at which relative service flow decays – and $T$ represents the durability of the good – the length of its useful life. The following proposition states that under some restrictions on preferences faster obsolescence, higher durability and higher price all make durable consumption more sensitive to changes in wealth.

**Proposition 1** Let $x_\alpha = g(T-\alpha)$, $\alpha = 0, \ldots, T$. Then

(i) An increase in the rate of obsolescence increases (decreases) $\theta$ if $u$ has decreasing (increasing) absolute risk aversion.

(ii) Assume that $u(c) = \frac{1}{1-\sigma}c^{1-\sigma}$ with $\sigma \geq 1$. Then $\theta$ increases in $p_0$.

(iii) Let $0 < \sigma \leq \sigma^* = 1.36$ and $\beta^T(1 + \beta) > 1$ (so (2) of Theorem 1 is satisfied for $T$ and $T+1$). Then, $\theta$ increases with durability (for any $g$ and $p_0$).

**Proof:** See Appendix 1.

The critical value $\sigma^* = 1.36$ has been computed numerically (and the proof explains how $\sigma^*$ is defined). A typical assumption on preferences is decreasing absolute risk aversion, which implies that $\theta$ increases with the rate of obsolescence.

Higher obsolescence rate makes the service flow decline more steeply with age. As a result, durables are replaced more frequently, on average, and this makes aggregate durable demand more wealth elastic. In contrast, extending the lifetime of durables does not shift class boundaries but changes the optimal replacement rule for consumers at the bottom of the wealth distribution (that used to follow the $(T+1)$-fixed rule).

Since we assumed that durables are only available in one size, a higher $p_0$ corresponds to a durable good that is more lumpy. When the durable is more expensive, the budgets corresponding to all fixed rules become larger. As a result, the kinks in $V_\alpha(b)$ become less pronounced, and the fixed rule classes become smaller in relation to flexible rule classes.

### 5.2 Lag times in consumption response

In response to a windfall, all fixed rule consumers immediately change their non-durable consumption by an amount equal to the annuity value of this windfall. Flexible rule consumers instead adjust their durable consumption, but only after a delay. The initial delay occurs because consumers synchronize replacement with new model introductions, so durable spending shows no response until the end of the current model cycle. Afterward, the adjustment of durable spending will be gradual and protracted, because flexible rule consumer save their marginal windfall until it can pay for replacing some future durable one period

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4 A small mass of consumers will change their consumption class as a result of change in wealth.
earlier. Depending on their flexible rule and their wealth, consumers choose different delays for spending their marginal windfall, which makes the adjustment of durable spending protracted.

On aggregate, it appears that initially the total consumption responds by less than the annuity value of the aggregate windfall, and that durable consumption responds with inertia. Both of these features find empirical support (e.g. Campbell and Deaton (1989), Caballero (1990)), they are common for \((s,S)\) models and do not depend on discrete obsolescence. The result that is special to our model is that adjustment of durable consumption has an extra lag until the new model becomes available.

5.3 Elasticity of demand for durables

The wealth intervals that correspond to fixed rule classes and flexible rule classes are interlaced. Therefore, elasticity of demand for the durable is non-monotonic in wealth. For example, the richest consumers in class \(C_1\) have zero elasticity of demand, because they already purchase every new model available. They are followed by consumers who use a 1-flexible rule and spend their windfalls only on durables, followed by \(C_2\) class with zero elasticity of demand and so on. The poorest consumers that follow the \(T+1\)-fixed rule (i.e. they never purchase the durable) exhibit zero elasticity.

Typically, one would expect that poorer consumers have a higher elasticity of demand, which is why producers try to target them with discounts. In our model, however, offering a price cut to the poorest consumers in order to boost sales is only marginally effective. The producers should instead target “frequency switchers” that use flexible rules. Among those, the class that shows the fastest response to a discount is consumers who follow a 1-flexible rule, because they make the most frequent replacements.

5.4 Optimal decision rule in an investment problem

Investment problems usually do not have budget constraints. We can eliminate the budget constraint from our model by assuming that \(u(c)\) is linear and that consumers can afford any replacement sequence. Then the investment problem becomes a special case of our model. To generate differences in optimal replacement rules, we assume that consumers have different preferences over the durable. In particular, we assume that a consumer of type \(h\) derive utility \(xh\) from a durable with the service flow \(x\). Then, his maximization problem is

\[
\max_{\{\tau_k, d_k\}} (vh - b)
\]

where \(v\) is the value of a durable purchasing policy and \(b\) is the budget. The following corollary to Theorem 2 shows that the optimal policy of any consumer is a fixed rule without delays.

**Corollary** For any \(h\), the optimal policy is a \(R^*\)-fixed rule without delays, where

\[
R^* = \arg \max_R \left( v_{T,R} \cdot h - b_{T,R} \right)
\]

**Proof:** See Appendix.
If we think of consumers as firms and consumer types as firm-specific productivities, then our model predicts that different firms coordinate their investment decisions with the obsolescence cycle. Investment is spiky not just on firm level, but on the industry level as well.

Note that flexible rules disappear in the model without the budget constraint because the utility function is linear in non-durable consumption, and therefore there is no incentive for consumption-smoothing.

6 Random Period Length

In this section, we solve the durable replacement problem with random arrival times. In our basic framework with certainty, the consumers can perfectly predict when new models will arrive to the market. By contrast, in a model with Poisson arrivals, the hazard rate stays constant, and the expected service flow of a new durable is independent of the time of purchase. Therefore, consumers have no incentive to time their purchases near the model introduction dates. We adopt a model that combines these two extreme cases. We assume that the development times of new models are i. i. d. random variables distributed on \([S, \infty)\), where \(S \geq 0\) is the minimum gestation period. If \(\tau\) is the time it takes to develop and introduce a new model into the market, then \(\tau - S\) has an exponential distribution with parameter \(\lambda\). Thus, the average arrival time is \(S + 1/\lambda\). Note that our deterministic model is the limit case when \(S = 1\) and \(\lambda \to \infty\), and that the pure Poisson arrival model corresponds to the case when \(S = 0\) and \(\lambda > 0\).

For tractability, we assume that there is only one good, the durable, and that each agent has a lifetime budget \(b\) to spend on durables. This model focuses exclusively on the intertemporal trade-off of purchasing the durable at different points in time, and excludes the possibility of an on-going substitution between the durable and other goods. Since we are primarily interested in the timing of durable purchases, this simplified specification seems appropriate.

In many durable goods markets, the quality-adjusted price for the durable falls over time because of manufacturing efficiency improvements. Thus, we now also let the price of the durable fall exponentially over time: \(p_t = p_0 e^{-\gamma t}\). With falling prices it may become attractive to buy a durable with delay: though its expected service flow decreases, the durable also becomes cheaper.

The state variables for the consumer are \(\alpha\) - the technological age of the endowed durable, \(\bar{b}_t = b_t/p_t\) - the “purchasing power” of the consumer, and \(s\) - the time since the last arrival of a new model (i.e. the age of the current model). The law of motion for the purchasing power is

\[
\bar{b}_{t+\Delta t} = (\bar{b}_t - \delta_t) e^{(r+\gamma)\Delta t},
\]

where, as before, \(\delta_t = 1\) if a new durable is purchased at date \(t\) and \(\delta_t = 0\) otherwise.

\(^5\)Because we assumed that durable budgets are exogenously determined, there is not an exact correspondence between the durable purchasing policy and the solution to an optimal consumption problem with two goods. In a dynamic optimization problem with two goods, the agents will optimally reallocate their budgets between durables and non-durables when new information becomes available. For example, if the design cycle has been unexpectedly long, an agent may want to start using part of his durable budget to increase his non-durable consumption.
Let \( V_\alpha (\bar{b}, s) \) be a consumer’s total discounted value of holding a purchasing power \( \bar{b} \) and a durable of technological age \( \alpha \) at the moment when a time \( s \) has elapsed since the last arrival of a new model. We restrict attention to the case \( T = 1 \), so \( \alpha \in \{0, 1\} \). This captures the main insights from the extended model without making the proofs excessively complicated.

When the new model reaches age \( S \), innovations start arriving at a constant Poisson rate \( \lambda \) and \( s \) becomes uninformative about the time of the next arrival. Therefore, the value function ceases to depend on \( s \):

\[
V_\alpha (\bar{b}, s) = V_\alpha (\bar{b}, S), \quad s \geq S.
\]

\( V_0 (\bar{b}, S) \) is the value of the service flow \( x_0 \) until the next Poisson event and the continuation value of an old model afterwards:

\[
V_0 (\bar{b}, S) = \int_0^\infty dt \lambda e^{-\lambda t} \left[ \int_0^t d\tau e^{-\rho \tau} x_0 + e^{-\rho t} V_1 (\bar{b} e^{(r+\gamma)t}, 0) \right] = \frac{x_0}{\lambda + \rho} + \int_0^\infty \lambda e^{-(\lambda + \rho)t} V_1 (\bar{b} e^{(r+\gamma)t}, 0) dt.
\]

(5)

For \( s < S \),

\[
V_0 (\bar{b}, s) = \int_0^{S-s} x_0 e^{-\rho \tau} d\tau + e^{-\rho (S-s)} V_0 (\bar{b} e^{(r+\gamma)(S-s)}, S).
\]

Replacement decision: The agent chooses the delay in replacement \( d (\bar{b}) \) of a depreciated good \( (x_1) \) since the last arrival of a new model. It is convenient to distinguish between two cases: \( d < S \) and \( d > S \), and separate the optimization problems over these intervals. Call the corresponding value functions \( V_1^L (\bar{b}) \) and \( V_1^R (\bar{b}) \). Both of those value functions are measured at the point where \( s = 0 \). Then

\[
V_1 (\bar{b}, 0) = \max \{ V_1^L (\bar{b}), V_1^R (\bar{b}) \},
\]

where

\[
V_1^L (\bar{b}) = \max_{0 \leq d \leq s} \left( \frac{x_1 (1 - e^{-\rho d})}{\rho} + e^{-\rho d} V_0 (\bar{b} e^{(r+\gamma)d} - 1, d) \right)
\]

(6)

and

\[
V_1^R (\bar{b}) = \max_{d \geq S} \int_0^S x_1 e^{-\rho \tau} d\tau + e^{-\rho S} \int_S^{S-d} dt \lambda e^{-\lambda t} \left[ \int_0^t x_1 e^{-\rho \tau} d\tau + e^{-\rho t} V_1 (\bar{b} e^{(r+\gamma)(t+S)}, 0) \right] + e^{-\lambda (d-S)} \int_S^d x_1 e^{-\rho \tau} d\tau + e^{-\rho d} V_0 (\bar{b} e^{(r+\gamma)d} - 1, S).
\]

(7)

The first term in the above expression is the value of holding the depreciated good until the new model reaches age \( S \), the second term is the expected value of holding the depreciated good between \( S \) and \( d \) and the third term is the expected value of the replacement at \( d \). (Note here that when a new good is bought at the time when \( s > S \), the corresponding continuation value of a new durable is as if it were of age \( S \).) The following proposition characterizes the optimal delay.

**Proposition 2:** Let \( \rho = r + \gamma \). Then there is an interval \([S, \bar{S}]\) with \( 0 \leq S < S < \bar{S} \) such that no durable purchases are made in the interval \([S, S]\) after new model arrivals. That is

\[
d (\bar{b}) \leq S \text{ or } d (\bar{b}) \leq \bar{S} \text{ for all } \bar{b} \geq 0.
\]
Figure 4: No-purchase intervals for various values of $\gamma$, $\lambda$ and $S$.

Moreover, all consumers who can afford the new model either purchase it right away (when $s = 0$) or after date $S$:

$$d\left(\bar{b}\right) = 0 \text{ or } d\left(\bar{b}\right) \geq \bar{S} \text{ for all } \bar{b} \geq 1.$$ 

**Proof:** See Appendix.

Thus, demand for the durable falls to zero as the date of possible new arrivals (that is, date $S$) draws sufficiently close and stays at zero for some time after date $S$. This result mirrors our Theorem 2, where we showed that new models are always purchased without delay for $r = \rho$ and $\gamma = 0$. By continuity, Proposition 2 should also hold when $r = \rho$ and $\gamma > 0$ is sufficiently small. We investigate numerically whether the no-purchase intervals are quantitatively important, especially when the price of the durable is falling.

For the simulations, we choose parameter values that we think are representative of markets for high-tech products. If $x_0$ and $x_1$ are the relative service flows of the durables in a detrended model, then the corresponding absolute service flows are $z_\tau = e^{x_\tau}$, $\tau = 0, 1$ (see the Appendix). Hence, $z_0/z_1 = e^{(x_0-x_1)}$ is the relative advantage of a new model. We set $x_0 = 0.4$ and $x_1 = 0$, which corresponds to a new model providing 50 percent more service than old models. We perform simulations for two average introduction times: $A = S + 1/\lambda = \{3, 5\}$ years. We set parameters $\rho = r = 0.04$ and experiment with different values of $\gamma$, $S$ and $\lambda$.

Figure 4 shows our results. In each panel, purchase delay is on the vertical axis and $\lambda$ is on the horizontal axis. The average introduction time $A$ is 3 years for the left panels and 5 years for the right panels, while $\gamma = 0.05$ for the top panels and $\gamma = 0.12$ for the bottom panels. Since in each panel the average introduction time is kept constant, as $\lambda$ varies, $S$ needs to be adjusted accordingly. Let $S(\lambda) = A - 1/\lambda$. This function is plotted as a thin solid line in each panel. The thick lines show the ends of the no-purchase interval, $S(\lambda)$ and...
\( \bar{S}(\lambda) \). In all cases, \( \underline{S}(\lambda) \leq S(\lambda) \leq \bar{S}(\lambda) \). All three lines cross at \( S = 0 \), which corresponds to the case of pure Poisson arrival times.

Note that when \( \gamma = 0.05 \), \( \underline{S} = 0 \). When the average introduction time is 3 (top-left panel), note also that \( \bar{S} > 3 \) for \( \lambda \geq 0.4 \). For those parameters, the time between new arrivals will often be shorter than \( \bar{S} \). That is, the consumers who wanted to delay the purchase beyond \( \bar{S} \) are frequently ‘surprised’ with a new arrival before the time when they were prepared to buy a new model. Obviously, when this happens, the consumers begin the new cycle with a higher purchasing power; and those with a high enough \( \bar{b} \) will buy the new model right away. That is, for some consumers, the surprise provokes an earlier purchase than what was ‘scheduled’. In reality, then, the majority of the purchases will be perfectly synchronized with the arrivals of new models, as in the deterministic setting. When \( \gamma = 0.12 \), \( \bar{S} > 0 \) for high values of \( \lambda \). Here, because prices are falling rapidly, some consumers are prepared to wait for a while after a new model arrives before purchasing it. So the initial purchase spike, produced by those consumers with \( d(\bar{b}) = 0 \), is followed by a period of positive demand (until \( \underline{S} \)). Afterwards, demand drops to zero and purchases resume only if the period turns out to be sufficiently long.

7 Conclusions

We have presented a model of durable goods that highlights the difference between obsolescence and physical wear and tear. The basic model is simple and it can be solved analytically. We identify discrete obsolescence as a distinct source of demand fluctuations, and explain how it affects the transmission of wealth shocks. The key implications of the basic model carry over to the case when obsolescence is stochastic and the relative price of the durable is falling.

Our model offers a building block for a general equilibrium analysis of an investment problem with capital obsolescence. Periodic obsolescence makes investment spiky even at the aggregate level, although interest rate adjustments will partially smooth out these spikes. Our framework can generate cyclical investment patterns and suggests a relationship between technological innovations in capital goods and the business cycle.
References


Appendix 1: Proofs

Detrending: Our model can be viewed as the detrended version of a fully dynamic model with a constant rate of technical progress. Suppose that a model $\tau$ provides a constant service flow $z_\tau$ for the duration of its useful life, in the interval $[\tau, \tau + T]$, and that $z_\tau = e^{g\tau}$, where $g$ is the rate of technical progress, or, equivalently, the rate of decrease of the quality-adjusted price for the durable. Now assume that the consumers’ utility function is $\hat{v}(z, c) = \ln(z) + u(c)$, where $z$ is the service flow of the durable good and $c$ is the flow of non-durable consumption. This dynamic model corresponds to the stationary model we propose when

$$x_\alpha = g(T - \alpha) \quad \text{for } \alpha = 0, \ldots, T.$$ 

Indeed, let $\alpha : \mathbb{R}_+ \to \{0, \ldots, T\}$ and $c : \mathbb{R}_+ \to \mathbb{R}_+$ be two measurable functions representing the consumption trajectory of a consumer (where $\alpha(t)$ is the technological age of the durable being consumed at time $t$). For any $r \in \mathbb{R}$, let $[r]$ denote the largest integer less than or equal to $r$. Note that along that trajectory, the model being consumed at time $t$ is $\tau(t) = [t] - \alpha(t)$. Thus, the total discounted utility for the trajectory $(\alpha, c)$ is

$$U(\alpha, c) = \int_0^\infty e^{-\rho t}[\ln(z_{\tau(t)}) + u(c(t))]dt = \int_0^\infty e^{-\rho t}[g([t] - T + T - \alpha(t)) + u(c(t))]dt$$

$$= K + \int_0^\infty e^{-\rho t}[x_{\alpha(t)} + u(c(t))]dt,$$

where

$$K = \int_0^\infty e^{-\rho t}g([t] - T)dt = \sum_{k=0}^\infty gk \int_0^1 e^{-\rho(k+1)}dt - \frac{gT}{\rho} = \frac{g}{\rho} \left[ \frac{e^{-\rho}}{1 - e^{-\rho}} - T \right].$$

Arbitrarily, we can re-normalize utility to set $K = 0$ without changing the consumer’s preferences over consumption paths. Then, the total discounted utility coincides with that of a consumer with utility function $v(\alpha, c) = x_\alpha + u(c)$.

Proof of Theorem 1: Suppose the agent is endowed with a durable of age $\alpha$ and follows an arbitrary purchasing policy $\tau = \{\tau_k\}_{k=1}^\infty$. We first show that the total cost and value $(b, v)$ of policy $\tau$ can be represented as a convex combination of the points $\{(b_{T,R}, v_{T,R})\}_{R=1}^{T+1}$. Let $\tau_0 = -\alpha$ and $r_k = \min \{\tau_{k+1} - \tau_k, T\}$ for all $k \geq 0$. Then

$$b = p_0 \sum_{k \geq 1} \beta^{r_k} \quad \text{and} \quad v = X_{\alpha, r_0} + \sum_{k \geq 1} \beta^{r_k} X_{0, r_k}.$$ 

Define $K_R = \{k \geq 1 \mid r_k = R\}$ for $R = 1, \ldots, T$. Then

$$\beta^{r_1} = [\beta^{r_1} - \beta^{r_2}] + [\beta^{r_2} - \beta^{r_3}] + \cdots \geq \sum_{R=1}^T \sum_{k \in K_R} \beta^{r_k} (1 - \beta^R),$$ 

where the inequality is strict if for some $k \in K_T$, $\tau_{k+1} - \tau_k > T$. Let $\lambda_R = \sum_{K_R} \beta^{r_k} (1 - \beta^R)$ for $R = 1, \ldots, T$, and let $\lambda_{T+1} = 1 - \sum_{R=1}^T \lambda_R$. Thus $\lambda_R \geq 0$ for all $R$, $\sum_{R=1}^{T+1} \lambda_R = 1$, and since $b_{T,T+1} = v_{T,T+1} = 0$. 

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Let $\tau$ be such a policy and let $v - X_{\alpha,r_0} = \sum_{R=1}^{T} X_{0,R} \beta^{\tau_k} = \sum_{R=1}^{T} X_{0,R} (1 - \beta^R) b_{T,R} = \sum_{R=1}^{T+1} \lambda_R b_{T,R}$.

Put differently, $(b, v - X_{\alpha,r_0}) = \sum_R \lambda_R (b_{T,R}, v_{T,R})$ is a convex combination of the two-dimensional vectors $(b_{T,R}, v_{T,R})$. Note that when $\alpha = T$, $X_{\alpha,r_0} = 0$ for all $r_0$.

We next deduce an optimal policy for the case where $\alpha = T$ (i.e., when the agent is endowed with a useless durable). If $b \geq b_{T,1}$, the agent can afford to replace the durable every period and $V_T(b) = v_{T,1}$ (moreover, if $b > b_{T,1}$, it is not possible for the agent to spend the budget $b$ in durables). For what follows assume that $b < b_{T,1}$. Let $R$ and $\lambda_R \in [0,1]$ be such that $b = \lambda_R^* b_R + (1 - \lambda_R^*) b_{R+1}$. Since $(b, V_T(b)) = \sum \lambda_R (b_{T,R}, v_{T,R})$ for some nonnegative weights $\lambda_R$ adding to 1, we have that $V_T(b) = \lambda_R^* v_{T,R} + (1 - \lambda_R^*) v_{T,R+1}$. To conclude, we only need to show that this bound is attained. For this we need to show that there exists a policy $\tau$ such that $\sum_{k \in K_R} \beta^{\tau_k} (1 - \beta^R) = \lambda_R^*$ and $\sum_{k \in K_R+1} \beta^{\tau_k} (1 - \beta^{R+1}) = 1 - \lambda_R^*$. Put differently, we need to show that there exists an $R$-flexible rule with budget $b$.

Assume that $R < T$ and let $B_R^*$ denote the set of budgets $b(\tau)$ corresponding to policies $\tau$ that are $R$-flexible rules and satisfy $\tau_1 = 0$ (that is, $\tau$ makes a purchase in the first period). Let $\tau$ be such a policy and $\tau'$ be its continuation policy from the period of the second purchase onward: $\tau'_t = \tau_{t+1} - \tau_1$ for all $t \geq 1$. Then, $\tau'$ is also an $R$-flexible rule and $\tau'_1 = 0$ and its corresponding budget $b(\tau') \in B_R^*$. Now, either $b(\tau) = p_0 + \beta^R b(\tau')$ (if $\tau_2 = R$) or $b(\tau) = p_0 + \beta^{R+1} b(\tau')$ (if $\tau_2 = R+1$). Therefore, $B_R^*$ is the largest set $B$ such that $B = [p_0 + \beta^R B] \cup [p_0 + \beta^{R+1} B]$. Observe that $p_0 + \beta^{R+1} b_{T,R+1} = b_{T,R+1}$ and $p_0 + \beta^R b_{T,R} = b_{T,R}$, and that $p_0 + \beta^R b_{T,R+1} < p_0 + \beta^{R+1} b_{T,R}$ when $\beta^{T-1} (1 + \beta) > 1$. Therefore $B = [b_{T,R+1}, b_{T,R}]$ is a fixed point of the above equation. Since $p_0 + \beta^R d < d$ for all $d > b_{T,R}$ and $p_0 + \beta^{R+1} d > d$ for all $d < b_{T,R+1}$, $B$ is also the largest such fixed point, and thus $B_R^* = B$. That is, for each budget $b \in B_R^*$, there exists a $(R, b)$-flexible rule (that spends the budget $b$ exactly). The proof for $R = T$ is similar (here $b_{T,T+1} = 0$ and we must consider policies $\tau$ where $\tau_{k+1} - \tau_k > T + 1$ for some $k$).

Finally, observe that if $(T, b)$ is the initial state and $\tau$ and $\tau$ are two $(R, b)$-flexible rules (they spend the same budget $b$), then their corresponding $\lambda_R$ (and $1 - \lambda_R$) must coincide, and therefore they must have the same value as well. In particular, if $b \in [b_{T,R+1}, b_{T,R}]$, then any $R$-flexible rule that spends the budget $b$ exactly is an optimal policy.

By construction, the value of following an $(R, b)$-flexible rule starting from a durable of age $T$ is given by

$$V_T(b) = v_{T,R+1} + A_R (b - b_{T,R+1}), b \in [b_{T,R+1}, b_{T,R}], R = T, \ldots, 1.$$
policy is an optimal policy for the subproblem that arises in the second period after following the policy in the first period.

If starting with a budget $b \in [p_0, b_{T,1}] = [b_{r+1,T+1}, b_{T,1}]$, a consumer buys a durable in the first period and then keeps it for the next $\alpha - 1$ periods, his budget at the beginning of period $\alpha \geq 1$ is $\theta_\alpha(b) = (b - p_0)/\beta^\alpha$. Moreover, for any $1 \leq R \leq T + 1$ and $1 \leq \alpha \leq \min \{R, T\}$, $\theta_\alpha(b_{T,R}) = b_{a,R}$.

Assume that the initial state is $(\alpha, b)$, where $1 \leq \alpha < T$ and $b \in [b_{a,R+1}, b_{a,R}]$ for some $\alpha \leq R \leq T$. Let $\bar{b} = p_0 + \beta^\alpha b$. Then $b = \theta_\alpha(\bar{b})$. Since $b \in [b_{a,R+1}, b_{a,R}]$, it must be that $b \in [b_{T,R+1}, b_{T,R}]$. Therefore, starting at state $(T, \bar{b})$, it is optimal to follow an $R$-flexible rule. Assume he does so. Then, after $\alpha$ periods his state becomes $(\alpha, b)$, and from state $(\alpha, b)$ he must be following an $R$-flexible rule as well. Hence, the agent must keep the durable for another $R - \alpha$ periods (at least). At that point, he arrives at state $(R, b/\beta^{R-\alpha})$. Note that $(1/\beta^{R-\alpha})[b_{a,R+1}, b_{a,R}] = [b_{r,R+1}, b_{r,R}]$ and that $\beta^{R-\alpha} b_{R+1,R+1} = b_{a+1,R+1} \in (b_{a,R+1}, b_{a,R})$. Hence, if $b/b^{R-\alpha} \in [b_{R,R+1}, b_{R,R+1, R+1})$ he must keep the durable this period and buy a new durable next period, so his continuation value is $\bar{x}_R + V_T(b/\beta^{R+1-\alpha})$. If $b/b^{R-\alpha} \in [b_{R,R+1}, b_{R,R}]$ he can optimally buy a new durable this period, and his continuation value is $V_T(b/\beta^{R-\alpha})$. Therefore

$$V_\alpha(b) = \begin{cases} X_{\alpha,R+1} + \frac{\beta^{R+1-\alpha} V_T(b/\beta^{R+1-\alpha})}{\beta^{R-\alpha}} & \text{for } b \in [b_{a,R+1}, b_{a+1,R+1}] \\ X_{\alpha,R} + \frac{\beta^{R-\alpha} V_T(b/\beta^{R-\alpha})}{\beta^{R-\alpha}} & \text{for } b \in [b_{a+1,R+1}, b_{a,R}] \end{cases}$$

Suppose that $b \in [b_{a,R+1}, b_{a+1,R+1}]$. Then $b/b^{R+1-\alpha} \in [b_{R,R+1}, b_{R,R+1, R+1}/\beta] \subset [b_{T,R,1}, b_{T,R}]$. Therefore, $V_T(b/b^{R+1-\alpha}) = v_{T,R+1} + A_R(b/b^{R+1-\alpha} - b_{T,R+1})$, and

$$X_{\alpha,R+1} + \frac{\beta^{R+1-\alpha} V_T(b/\beta^{R+1-\alpha})}{\beta^{R-\alpha}} = v_{a,R+1} + A_R(b - b_{a,R+1}).$$

Now suppose that $b \in [b_{a+1,R+1}, b_{a,R}]$. Then $b/b^{R-\alpha} \in [b_{R,R+1}, b_{R,R}/\beta] \subset [b_{T,R,1}, b_{T,R}]$. Therefore, $V_T(b/b^{R-\alpha}) = v_{T,R+1} + A_R\left(b/b^{R-\alpha} - b_{T,R+1}\right)$, and tedious algebra shows again that

$$X_{\alpha,R} + \frac{\beta^{R-\alpha} V_T(b/\beta^{R-\alpha})}{\beta^{R-\alpha}} = v_{a,R+1} + A_R(b - b_{a,R+1}).$$

Therefore, for all $\alpha \leq R \leq T$ and $b \in [b_{a,R+1}, b_{a,R}]$, $V_\alpha(b) = v_{a,R+1} + A_R(b - b_{a,R+1})$. It remains to find $V_\alpha(b)$ for $b > b_{a,\alpha}$. We claim that $V_\alpha(b) = V_T(b)$ for all $b > b_{a,\alpha}$. Since $b_{a,R} = b_{T,R}$ for all $R \leq \alpha$, we have that $V_T(b) = v_{a,R+1} + A_R(b - b_{a,R+1})$ for all $b \in [b_{a,R+1}, b_{a,R}]$ and $1 \leq R < \alpha$, and the claim would complete the proof. To prove our claim, we show that

$$V_T(b) > X_{a,s+\alpha} + \beta^s V_T(b/\beta^s)$$

for all $s > 0$ and $b > b_{a,\alpha}$.

That is, when $b > b_{a,\alpha}$, the consumer strictly prefers to replace the durable immediately than to replace it at any later time. One can check that the above inequality holds when $b = b_{a,\alpha}$. Also, since $V_T$ is concave, the function $V_T(b)$ has a higher slope than the function on the right hand side for any $b > 0$. Hence, the inequality holds for every $b > b_{a,\alpha}$.

**Proof of Theorem 2:** Consider an arbitrary purchasing policy \{$(\tau'_{k}, d_k)$\}, where $\tau'_{k} + d_k$ denotes the time of the $k$-th purchase and $\tau'_{k} \in \mathbb{N}$ its corresponding period (so $d_k \in [0,1)$ denotes its “delay”). Let $r_0 = \alpha + \tau'_{1}$, and for all $k \geq 1$, let $\tau_{k} = \tau'_{k} - \tau'_{1}$ and $r_k = \min \{\tau_{k+1} - \tau_k, T\}$. Then the continuation budget and value of such a policy, at the
beginning of period $\tau_1$, are respectively

$$b = p_0 \sum_{k \geq 1} \beta^k e^{-\rho dk} = p_0 \sum_{k \geq 1} \beta^k - p_0 \sum_{k \geq 1} \beta^k (1 - e^{-\rho dk})$$
$$v = \sum_{k \geq 1} \beta^k X_{0,r_k} - \sum_{k \geq 1} \beta^k (x_0 - x_{r_{k-1}})(1 - e^{-\rho dk})/\rho.$$ 

Let $I_{r_0} = 1$ and $I_R = 0$ for $R \neq r_0$. For $1 \leq R \leq T$, let $K_R = \{k \geq 1 \mid r_k = R\}$,

$$\lambda_R = \sum_{k \in K_R} \beta^k (1 - \beta^R), \quad \bar{\gamma}_R = I_R \beta^0 + \sum_{k \in K_R} \beta^k + 1$$

and

$$\gamma_R = I_R \beta^0 \left[1 - \frac{1 - e^{-\rho d_1}}{1 - \beta}\right] + \sum_{k \in K_R} \beta^k \left[1 - \frac{1 - e^{-\rho d_{k+1}}}{1 - \beta}\right], \quad \text{(8)}$$

so that

$$[b] = [\hat{b}] - \sum_{R=1}^T \gamma_R \left[p_0 (1 - \beta) \tilde{x}_0 - \tilde{x}_R\right], \quad \text{where} \quad [\hat{b}] = \sum_{R=1}^T \lambda_R \left[b_{T,R} \hat{v}_{T,R}\right].$$

The coefficient $\lambda_R$ incorporates the discounting of all the periods in which a purchase is made for a good that will be replaced at age $R$. By contrast, $\gamma_R$ incorporates the discounting of all the periods in which a purchase is made to replace a good of age $R$. The adjustment, reflected in the factor multiplying $I_{r_0}$, of $\gamma_{r_0}$ (and $\bar{\gamma}_{r_0}$) is required to take into account the first purchase that replaces the endowed good (that in our accounting, was not previously purchased). Observe that for $1 \leq R \leq T - 1$, $k \in K_R$ implies that $\tau_{k+1} = \tau_k + R$ (if $k \in R_T$ then $\tau_{k+1} \geq \tau_k + T$, where strict inequality holds when a useless good is not replaced for one or more periods).

Therefore

$$\bar{\gamma}_R = I_R + \left[\frac{\beta^R}{1 - \beta^R}\right] \lambda_R \quad \text{for all } 1 \leq R \leq T - 1,$$

$$\sum_{R=1}^T \lambda_R \left[\frac{\beta^R}{1 - \beta^R}\right] = \sum_{k \geq 1} \beta^k = \sum_{R=1}^T \bar{\gamma}_R = \sum_{R=1}^{T-1} \left[\frac{\beta^R}{1 - \beta^R} \lambda_R + I_R\right] + \bar{\gamma}_T.$$

Hence, $\bar{\gamma}_T = \sum_{R=1}^{T-1} \lambda_R + \lambda_T/(1 - \beta^T) + I_T - 1 = I_T + \lambda_T \beta^T/(1 - \beta^T)$. Let $\Lambda = \{\lambda \in \mathbb{R}_+^T \mid \sum_{R=1}^T \lambda_R \leq 1\}$, and

$$\Gamma = \{(\lambda, \gamma) \in \Lambda \times \mathbb{R}_+^T \mid \gamma_R < \bar{\gamma}_R \quad \text{for} \ 1 \leq R \leq T\}.$$

CLAIM 1: Let $\{(\tau'_k, d_k)\}$ be an arbitrary purchasing policy and $(\lambda, \gamma)$ be the weights defined by (8). Then $(\lambda, \gamma) \in \Gamma$. Conversely, for any $(\lambda, \gamma) \in \Gamma$ (and $\tau'_1 \geq 1$), there exists a purchasing policy $\{(\tau'_k, d_k)\}$ that satisfies (8). Though this policy is usually not unique, all such policies have the same budget and value. Thus, with abuse of notation we will also refer to a $(\lambda, \gamma) \in \Gamma$ as a purchasing policy.

The argument above essentially contains the proof of this claim.

CLAIM 2: Suppose that the policy corresponds to an $R$-flexible rule where $\tau_1 = 0$ and the replacement of durables of age $R + 1$ is never delayed but the replacement of durables of
age \( R \) is sometimes delayed. Then, the policy is suboptimal: there exists another \( R \)-flexible rule without delays that costs the same and has a strictly higher value.

Proof: For such a policy, \( \lambda_R + \lambda_{R+1} = 1 \), \( \gamma_R > 0 \), \( \gamma_{R+1} = 0 \), and \( \lambda_k = \gamma_k = 0 \) for all \( k \notin \{ R, R+1 \} \). Moreover, since \( \gamma_R < \lambda_R \beta^R / (1 - \beta^R) \), we also have \( \lambda_R > 0 \). In this case, \((\hat{b}, \hat{v})\) is on the “Pareto frontier” (i.e., \( \hat{v} = V_T(\hat{b}) \)). The vector \( (b, v) = (p_0(1 - \beta), \hat{x}_0 - \hat{x}_R) \) has “slope” \( \sigma = [\hat{x}_0 - \hat{x}_R] / [p_0(1 - \beta)] \), and

\[
A_R = \frac{1}{p_0} \left[ X_{0,R} - \hat{x}_R \frac{1 - \beta^R}{1 - \beta} \right] \leq \left( 1 - \beta^R \right) \frac{\hat{x}_0 - \hat{x}_R}{p_0(1 - \beta)} < \sigma.
\]

So, as the delays increase (\( \gamma_R \) increases), \((b, v)\) moves away of \((\hat{b}, \hat{v})\), below the Pareto frontier. But, if \( \sigma < A_{R+1} \), the delays may eventually take \((b, v)\) back above the Pareto frontier. This could happen only if \( b < b_{T,R+1} \). But even if every durable of age \( R \) is replaced with delay, the cost of the policy is more than replacing the durables at age \( R+1 \) all the time. That is, \( b \geq b_{T,R+1} \). Therefore \( b_{T,R+1} \leq b \leq b_{T,R} \) and \( v < V_T(b) \), and there exists another \( R \)-flexible rule with no delays that costs \( b \) and has value \( V_T(b) \).

CLAIM 3: Suppose that the policy \( \{ (\tau_k, d_k) \} \) is such that \( \gamma_R > 0 \) for some \( R \). Then the policy is suboptimal: there exists another policy without delays that uses the same budget but has strictly higher value.

Assume that the policy has delays. We now recursively modify the policy by eliminating delays while maintaining the same budget and improving its value in every step. Let \( h = \lambda_1 + \lambda_2 \), \( \lambda_k = \lambda_k / h \) for \( k = 1, 2 \), and \( \hat{\gamma}_1 = \gamma_1 / h \). Then

\[
\begin{bmatrix} b \\ v \end{bmatrix} = h \begin{bmatrix} b_{T,1} \\ v_{T,1} \end{bmatrix} + \hat{\lambda}_2 \begin{bmatrix} b_{T,2} \\ v_{T,2} \end{bmatrix} - \hat{\gamma}_1 \begin{bmatrix} p_0(1 - \beta) \\ \hat{x}_0 - \hat{x}_1 \end{bmatrix} + \sum_{R=2}^{T} \lambda_R \begin{bmatrix} b_{T,R} \\ v_{T,R} \end{bmatrix} - \gamma_R \begin{bmatrix} p_0(1 - \beta) \\ \hat{x}_0 - \hat{x}_R \end{bmatrix}.
\]

The weights \( (\hat{\lambda}_1, \hat{\lambda}_2, \hat{\gamma}_1) \) represent a 1-flexible rule with delays (and \( \hat{\lambda}_1 + \hat{\lambda}_2 = 1 \)). If \( \gamma_1 > 0 \) (so \( \hat{\gamma}_1 > 0 \)), then by Claim 3 there exists another 1-flexible rule with weights \( (\hat{\lambda}_1, \hat{\lambda}_2, 0) \) that is better. Let \( \lambda'_k = h \hat{\lambda}_k \) for \( k = 1, 2 \), \( \gamma'_k = \hat{\gamma}_1 = 0 \), \( \lambda'_k = \lambda_k \) for \( k \geq 3 \), and \( \gamma'_k = \gamma_k \) for \( k \geq 2 \). The policy \((\lambda'_1, \gamma'_1)\) is better than the policy \((\lambda, \gamma)\) and has \( \gamma'_1 = 0 \). Now, let \( h = \lambda'_1 + \lambda'_2 \), \( \hat{\lambda}_k = \lambda_k / h \) for \( k = 2, 3 \), and \( \hat{\gamma}_2 = \gamma'_2 / h \). The weights \( (\hat{\lambda}_2, \hat{\lambda}_3, \hat{\gamma}_2) \) represent a 2-flexible rule with delays. Again, if \( \hat{\gamma}_2 > 0 \), Claim 3 implies that there exists a better 2-flexible rule without delays that can be used to modify \((\lambda', \gamma')\) and construct a new policy \((\lambda'', \gamma'')\) that is better, uses the same budget, and has \( \gamma''_1 = \gamma''_2 = 0 \). Continuing this way, after \( T \) steps, we will have constructed a policy \((\lambda^*, \gamma^*)\) with \( \gamma^* = 0 \), that uses the same budget and has a better value than \((\lambda, \gamma)\).

Finally, by Claim 2 (or Theorem 1), for any \( \tau'_1 \geq 1 \) and any weights \( \lambda^* \), there exist \( R \) and an \( R \)-flexible rule that uses the same budget \( \beta^{\tau'_1} b = \beta^{\tau'_1} \sum_k \lambda'_k b_{T,k} \) (from period 0 onward) and delivers a (weakly) better value. Therefore, the optimal value function \( V_T \) for the continuous-time economy coincides with that for the discrete-time economy (as defined in Theorem 1).

Proof of Corollary to Theorem 2: From Theorem 2, we can write

\[
v h - b = X_{o,r_0} h + \sum_{R=1}^{T} \lambda_R (v_{T,R} \cdot h - b_{T,R}) + \sum_{R=1}^{T} \gamma_R (p_0(1 - \beta) - h \cdot (\hat{x}_0 - \hat{x}_R)),
\]
where \( r_0 = \alpha + \tau'_1 \) and \( \tau'_1 \) is the period of the first purchase. Since the consumer chooses \( r_0 \geq \alpha \) and \( (\lambda, \gamma) \in \Gamma \) (where \( \Gamma \) was defined in the proof of Theorem 2), and the objective function is linear in \( (\lambda, \gamma) \), clearly \( \lambda_R \in \{0, 1\} \) and \( \gamma_R \in \{0, \gamma_R\} \). Since \( \sum \lambda_R \leq 1 \), \( \lambda_R = 1 \) implies \( \lambda_k = 0 \) for all \( k \neq R \), and consequently \( \gamma_k = 0 \) for all \( k \neq R \) as well. So, in the optimal solution, one and only one replacement frequency \( R \) is used, and we only need to show that the corresponding \( \gamma_R = 0 \).

Suppose that \( \lambda_R = 1 \). Then type \( h \) must prefer the \( R \)-fixed rule to any other fixed rule. In particular, he must prefer it over the \( R+1 \)-fixed rule: \( v_{T,R} \cdot h - b_{T,R} \geq v_{T,R+1} \cdot h - b_{T,R+1} \). This implies that

\[
h \geq \frac{b_{T,R} - b_{T,R+1}}{v_{T,R} - v_{T,R+1}} = \frac{1}{A_R}
\]

For \( \gamma_R = 0 \), it must be the case that \( p_0(1 - \beta) < h(\hat{x}_0 - \hat{x}_R) \). We show that this inequality holds when \( h = 1/A_R \), and therefore it also holds for any \( h \geq 1/A_R \). Indeed

\[
p_0(1 - \beta) < \frac{1}{A_R} (\hat{x}_0 - \hat{x}_R) \iff A_R p_0 \frac{1}{\hat{x}_0 - \hat{x}_R} < \frac{1}{1 - \beta} \quad \text{and}
\]

\[
A_R p_0 \frac{1}{\hat{x}_0 - \hat{x}_R} = \sum_{i=0}^{R-1} \beta^i \frac{\hat{x}_i - \hat{x}_R}{\hat{x}_0 - \hat{x}_R} < \sum_{i=0}^{\infty} \beta^i < \sum_{i=0}^{\infty} \beta^i = \frac{1}{1 - \beta}.
\]

Thus, given the choice of \( r_0 \), the optimal policy is a fixed rule without delays. To conclude, we need to show that \( r_0 \) is also consistent with this rule. That is, for all \( h \in \left[\frac{1}{A_R}, \frac{1}{A_{R-1}}\right] \) (i.e. all consumer types that choose the \( R \)-fixed rule), \( r_0 = R \). Let \( J(h) = v_{T,R} \cdot h - b_{T,R} \) be the continuation value of the optimal policy from the first purchase onward. We must show that if the good is of age \( R - 1 \) (or less), holding it for one more period is better than replacing it right away, and the opposite is true if the good is of age \( R \) (or older):

\[
\hat{x}_{R-1}h + \beta J(h) \geq J(h),
\]

\[
\hat{x}_Rh + \beta J(h) \leq J(h).
\]

It suffices to show that the first inequality holds for \( h = 1/A_{R-1} \) and the second one holds for \( h = 1/A_R \), that is

\[
A_{R-1} b_{T,R} \geq v_{T,R} - \frac{\hat{x}_{R-1}}{1 - \beta};
\]

\[
A_R b_{T,R} \leq v_{T,R} - \frac{\hat{x}_R}{1 - \beta}.
\]

After some algebra, both expressions above are actually equalities, which completes the proof.

**Proof of Theorem 3:** Recall that we defined \( c_0 = \bar{w}(1 - \beta) - p_0 \) and \( c_{T+1} = w(1 - \beta) \), so that \( w_{1,1}(c_0) = \bar{w} \) and \( w_{T,T+1}(c_{T+1}) = w \).

Let \( B(w, c) = w - c/(1 - \beta) \) be the budget left for durables when the total wealth is \( w \) and the agent consumes a constant per period budget \( c \) on non-durables. For fixed \( \alpha \) and \( w \), the function \( \varphi(c) = \hat{u}(c)/(1 - \beta) + V_{\alpha}(B(w, c)) \) is concave. Thus \( \hat{c} \) maximizes \( \varphi(c) \) if and only if \( 0 \in \partial \varphi(\hat{c}) \) (that is, 0 is a subdifferential of \( \varphi \) at \( \hat{c} \)) or equivalently, if and only if
$\hat{u}'(\hat{c}) \in \partial V_\alpha(B(w, \hat{c}))$. There are two cases corresponding to the situations where (1) $V_\alpha$ is differentiable at $B(w, \hat{c})$; and (2) $V_\alpha$ has a kink at $B(w, \hat{c})$.

**Case 1:** Observe that $B(w, c_R) \in (b_{\alpha+1}, b_\alpha)$ if and only if $w \in (w_{\alpha,R+1}(c_R), w_{\alpha,R}(c_R))$. Now, if $B(w, c_R) \in (b_{\alpha+1}, b_\alpha)$ for some $R$, then $\hat{u}'(c_R) = A_R = V_\alpha'(B(w, c_R))$, and $c_R$ is the optimal solution of problem (1). That is, when $w \in (w_{\alpha,R+1}(c_R), w_{\alpha,R}(c_R))$, it is optimal to consume a constant flow $c_R$ of non-durables and follow an $R$-flexible purchasing rule for the durable good. One can check that $B(w, c_\alpha) = b_{\alpha+1,\alpha+1} \iff w = w_{\alpha+1,\alpha+1}(c_\alpha)$ and $B(w, c_\alpha) = b_{\alpha-1,\alpha} \iff w = w_{\alpha-1,\alpha}(c_\alpha)$, and $w_{\alpha+1,\alpha}(c_\alpha) < w_{\alpha-1,\alpha}(c_\alpha) < w_{\alpha,\alpha}(c_\alpha)$. Therefore, $\delta^*(w)$ is given by (4).

**Case 2:** Observe that $A_{R-1} \leq \hat{u}'(c_{\alpha,R}(w)) \leq A_R$ if and only if $c_R \leq c_{\alpha,R}(w) \leq c_{R-1}$, or alternatively, if and only if $w \in [w_{\alpha,R}(c_\alpha), w_{\alpha,R}(c_{R-1})]$. Since $B(w, c_{\alpha,R}(w)) = b_{\alpha,R}$ and $\partial V_\alpha(b_{\alpha,R}) = [A_{R-1}, A_R]$, if $\hat{u}'(c_{\alpha,R}(w)) \in [A_{R-1}, A_R]$ for some $R$, then $c_{\alpha,R}(w)$ is the optimal solution of problem (1). That is, it is optimal to consume a constant flow $c_{\alpha,R}(w)$ of non-durables and follow the $R$-fixed purchasing rule for the durable good. In particular, $\delta^*(w) = 1$ if $R \leq \alpha$ (or equivalently, if $w \geq w_{\alpha,\alpha}(c_\alpha)$) and $\delta^*(w) = 0$ if $R > \alpha$, as stated in (4).

For a fixed $\alpha$, the intervals corresponding to case 1 alternate with those corresponding to case 2. Moreover, collectively, they are mutually exclusive and cover the whole wealth range.

**Proof of Proposition 1:**

(i) Since $w_{\alpha,R}(c_{R-1}) - w_{\alpha,R}(c_R) = (c_{R-1} - c_R)/(1 - \beta)$, the total size of the fixed-rule classes is

$$\varphi_T = \frac{1}{1 - \beta} \sum_{R=1}^{T} R[c_{R-1} - c_R] + \frac{c_T}{1 - \beta} - w = \frac{1}{1 - \beta} \sum_{R=1}^{T-1} [c_R - c_T] + \bar{w} - w - \frac{p_0}{1 - \beta}.$$  

(Recall that $c_0 = (1 - \beta)\bar{w} - p_0$ and $c_{T+1} = (1 - \beta)c_T$.) Similarly, since $w_{\alpha,R}(c_R) - w_{\alpha,R+1}(c_R) = b_{\alpha,R} - b_{\alpha,R+1}$, we have that $\mu(C_{R,R+1}) = p_0/(1 - \beta R)$ for $1 \leq R \leq T - 1$, and $\mu(C_{T,T+1}) = p_0/(1 - \beta)$. Therefore, the total size of the flexible-rule classes is

$$\psi_T = \sum_{R=1}^{T-1} \frac{p_0}{1 - \beta R} + \frac{p_0}{1 - \beta}.$$  

Note that $\theta = \psi_T/[\psi_T + \varphi_T]$ and that $\psi_T$ does not depend on $g$. Therefore,

$$\frac{\partial \theta}{\partial g} = \frac{-\psi_T}{(\psi_T + \varphi_T)^2} \frac{\partial \varphi_T}{\partial g} \quad \text{and} \quad \frac{\partial \varphi_T}{\partial g} = \frac{1}{1 - \beta} \sum_{R=1}^{T-1} \left( \frac{\partial c_R}{\partial g} - \frac{\partial c_T}{\partial g} \right).$$

A simple computation shows that

$$A_R = \frac{1}{p_0} \sum_{R=0}^{R-1} \beta^\alpha (\bar{x}_\alpha - \bar{x}_R) = g \left[ \frac{1 - \beta}{\rho p_0} \right] \sum_{R=0}^{R-1} \beta^\alpha (R - \alpha).$$

Since $\hat{u}'(c_R) = A_R$,

$$\frac{\partial c_R}{\partial g} = \frac{1}{\hat{u}''(c_R)} \frac{\partial A_R}{\partial g} = \frac{1}{\hat{u}''(c_R)} \frac{A_R}{g} = \frac{1}{\hat{u}''(c_R)}.$$
If \( u \) has decreasing (increasing) absolute risk aversion then so does \( \hat{u} \), and since \( c_1 > c_2 > \cdots > c_T \), for every \( R = 1, \ldots, T - 1, \)

\[
\frac{\partial c_R}{\partial g} - \frac{\partial c_T}{\partial g} = \frac{1}{g} \left[ \frac{\hat{u}'(c_R)}{\hat{u}''(c_R)} - \frac{\hat{u}'(c_T)}{\hat{u}''(c_T)} \right] < 0 \ (> 0).
\]

(ii) Let \( u(c) = c^{1-\sigma}/(1 - \sigma) \) with \( \sigma \geq 1 \). Then \( \hat{u}(c) = [(1 - \beta)/\rho]^{\sigma}[c^{1-\sigma}/(1 - \sigma)] \) and \( c_R = (1 - \beta)A_R^{-1/\sigma}/\rho \). Differentiating \( c_R \) with respect to \( p_0 \) gives

\[
\frac{\partial c_R}{\partial p_0} = -\frac{1}{\sigma} A_R \frac{\partial A_R}{\partial p_0} = \frac{1}{\sigma} c_R,
\]

so that

\[
\frac{\partial \varphi_T}{\partial p_0} = \frac{1}{\sigma p_0} \left( \frac{1}{1 - \beta} \sum_{R=1}^{T-1} [c_R - c_T] - \frac{p_0\sigma}{1 - \beta} \right) = \frac{1}{\sigma p_0} \left( \varphi_T - (\bar{w} - \hat{w}) - \frac{p_0}{1 - \beta} (\sigma - 1) \right)
\]

Then

\[
\frac{1}{\varphi_T} \frac{\partial \varphi_T}{\partial p_0} < \frac{1}{p_0} \frac{\partial \psi_T}{\partial p_0}
\]

The size of fixed rule classes grows at a slower rate than the size of flexible rule classes, hence

\[
\frac{\partial \theta}{\partial p_0} > 0.
\]

(iii) Let \( x_\alpha = g(T - \alpha) \) for \( \alpha = 1, \ldots, T \) and \( x'_\alpha = g(T' - \alpha) \) for \( \alpha = 1, \ldots, T' \), where \( T < T' \). Then, for all \( R = 1, \ldots, T, \)

\[
A_R = \frac{1}{p_0} \sum_{\alpha=0}^{R-1} \beta^\alpha (\hat{x}_\alpha - \hat{x}_R) = \frac{1}{p_0} \sum_{\alpha=0}^{R-1} \beta^\alpha (\hat{x}'_\alpha - \hat{x}'_R) = A'_R.
\]

This implies that the economy where durables last \( T \) periods and the economy where durables last \( T' > T \) periods have identical consumption levels \( c_1, \ldots, c_T \).

Clearly, \( \theta \) is increasing in \( T \) if and only if

\[
\frac{\varphi_{T+1}}{\varphi_T} < \frac{\psi_{T+1}}{\psi_T}.
\]

For all consumption classes to be non-empty, the interval \([\bar{w}, \hat{w}]\) must be such that \( \bar{w}(1 - \beta) > c_1 + p_0 \) and \( \bar{w}(1 - \beta) < c_T \). We now set \( \bar{w} = 0 \), in order to guarantee that class \( C_{T+1} \) is non-empty for all \( T \). Inequality (9) is harder to satisfy for smaller values of \( \bar{w} \). Therefore, we set \( \bar{w} = (c_1 + p_0)/(1 - \beta) \). If inequality (9) is satisfied for this \( \bar{w} \) then it must also hold for any larger \( \bar{w} \). Since \( \hat{u}(c) = [(1 - \beta)/\rho]^{\sigma}[c^{1-\sigma}/(1 - \sigma)] \), \( c_R = (1 - \beta)A_R^{-1/\sigma}/\rho \). The left hand side of (9) is then

\[
\frac{\varphi_{T+1}}{\varphi_T} = \frac{A_1^{-1/\sigma} + \sum_{R=1}^{T} \left( A_R^{-1/\sigma} - A_T^{-1/\sigma} \right)}{A_1^{-1/\sigma} + \sum_{R=1}^{T-1} \left( A_R^{-1/\sigma} - A_{T-1}^{-1/\sigma} \right)}.
\]
Noted that since $A_R$ is proportional to $g$, the above expression does not depend on $g$, and is only a function of $\beta$ and $\sigma$. Similarly, $\psi_{T+1}/\psi_T$ depends only on $\beta$. Numerical computations shows that there exists $\hat{\beta}(\beta) > 0$ such that (9) holds for all $\sigma \in (0, \hat{\beta}(\beta)]$. If we let $\sigma^* = \min_\beta \hat{\beta}(\beta) = 1.36$, then, independent of $\beta$, $\theta$ is increasing in $T$. ■

**Proof of Proposition 2**: We first construct and upper bound $W(\bar{b})$ for $V_1(\bar{b}, 0)$. Let $d$ be such that $e^{-\rho d} \bar{b} = 1$. The agent has to wait at least until $d$ before he can afford to purchase a new durable. If the agent consumes an old durable until $d$ and a new durable forever after, his total discounted value is

$$W(\bar{b}) = \frac{x_1}{\rho} (1 - e^{-\rho d}) + \frac{x_0}{\rho} e^{-\rho d} = \frac{x_1}{\rho} (1 - \bar{b}) + \frac{x_0}{\rho} \bar{b}.$$ 

Clearly, $V_1(\bar{b}, 0) \leq W(\bar{b})$ for all $\bar{b} \in [0, 1)$. Also, $V_1(0, 0) = W(0) = x_1/\rho$. Therefore

$$\frac{\partial}{\partial \bar{b}} V_1(0, 0) \leq W'(0) = \frac{x_0 - x_1}{\rho}.$$ 

Differentiating both sides of the Bellman equation (5) with respect to $\bar{b}$ and evaluating the derivative at $\bar{b} = 0$ gives

$$\frac{\partial}{\partial \bar{b}} V_0(0, S) = \frac{\partial}{\partial \bar{b}} V_1(0, 0).$$

For convenience, let $G (\bar{b}, d)$ denote the right hand side of (6). This function is defined for all $(\bar{b}, d)$ where a purchase with delay $d \leq S$ is feasible. That is, when $\bar{b} \geq e^{-(r + \gamma) S}$ and $\max \left\{ 0, \frac{1}{r + \gamma} \ln \left( \frac{1}{\bar{b}} \right) \right\} \leq d \leq S$. Now we show that when $\rho = r + \gamma$, $G (\bar{b}, d)$ is strictly decreasing in $d$ for all $\bar{b} \geq e^{-(r + \gamma) S}$. Differentiating $G$ with respect to $d$ and setting $\rho = r + \gamma$, we get

$$\frac{\partial G}{\partial d} (\bar{b}, d) = e^{-\rho d} \left[ \frac{\partial}{\partial \bar{b}} V_0 \left( \frac{e^{\rho S}}{\rho} (\bar{b} - e^{-\rho d} S), S \right) - \frac{(x_0 - x_1)}{\rho} \right]$$

$$< e^{-\rho d} \left[ \frac{\partial}{\partial \bar{b}} V_0 (0, S) - \frac{(x_0 - x_1)}{\rho} \right] \leq 0.$$ 

If $\bar{b} \geq 1$, any delay $d \in [0, S]$ is possible. Since $G (\bar{b}, d)$ is strictly decreasing in $d$ for all $d \in [0, S]$, the optimal delay must be either $d = 0$ or $d > S$.

If $\bar{b} < 1$, delays $d < \frac{1}{r + \gamma} \ln \left( \frac{1}{\bar{b}} \right)$ are not feasible, but since $G (\bar{b}, d)$ is decreasing in $d$, the maximum is either $d (\bar{b}) = \frac{1}{r + \gamma} \ln \left( \frac{1}{\bar{b}} \right) < S$ or $d (\bar{b}) > S$. ■

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