In this lecture we talk about how to value different options.

We will value options by assuming that no arbitrage opportunities exist.

Remember put-call parity.

To value put options, we just have to value call options and apply PC parity.

What determines option values?
- stock price, S (C goes up with S)
- strike price, X (C goes down with X)
- volatility, σ (C goes up with σ)
- time to maturity, T (C goes up with T)
- interest rate, r_f (C goes up with r_f)
- dividends, D (C goes down with D)

Why do these relations hold?

Restrictions on option values:
- C > 0 - why?
  - If a call expires in the money, you can replicate its payoff by buying 1 share of stock and borrowing PV(X + D).
  - The payoff to this would be (S_T - X).
  - This strategy has downside risk.
  - Thus, C > S_0 - PV(X) - PV(D).
- C < S_0 - why?
When does a call holder want to exercise?
- First consider a call on a stock with \( D = 0 \).
- There are 2 ways to reverse a call position:
  - exercise early
  - sell the option
- Since \( C > [S_0 - PV(X)] > (S_0 - X) \), it is never optimal to exercise an American call early.

If \( D = 0 \) and you want to reverse your position, sell your call but don’t exercise.
- European \( C \) = American \( C \) when \( D = 0 \).
- What if the stock pays a dividend?
- What if it pays a liquidating dividend?
- If \( D \) is high enough, it can be optimal to exercise your call option early.

Most option pricing is done with a binomial “tree” model.
- In a binomial tree, we assume that prices either go up or go down.
- For example, suppose we knew that the price of $100 stock would either go to $50 or $200 by year-end.
A call option with strike price $125 would have a payoff structure like:

\[
\begin{array}{c}
\text{C} \\
75 \\
0
\end{array}
\]

We can replicate the call option with a position in the stock and in 8% T-bills.

Suppose, for example, that you buy 1/2 share of the stock and borrow $23.15.

Your payoff at year-end will depend on \( S_T \).

If \( S_T = 50 \) then it is \( 25 - 1.08 \times 23.15 = 0 \).

If \( S_T = 200 \) then it is \( 100 - 25 = 75 \).

This position costs you \( 50 - 23.15 = 26.85 \).

Since this strategy costs you $26.85 today and its payoff is equal to the call's payoff, the call must cost $26.85.

Why is this true?

This is basically how “trees” work.

Binomial trees construct a riskless position out of stock, options and “cash.”

The payoff of the options plus cash is perfectly correlated with the stock price.

In the example, holding 1/2 share and writing 1 call produces a risk-free position.

The payoff to this position is always $25.

It costs you $26.85 today to get $25 at \( T \).

Suppose \( C = 30 \) (not 26.85), \( r_f = 8\% \).

We can make an arbitrage chart:

<table>
<thead>
<tr>
<th>Position</th>
<th>Initial CF</th>
<th>( S = 50 )</th>
<th>( S = 200 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Write 2 C</td>
<td>60</td>
<td>0</td>
<td>-150</td>
</tr>
<tr>
<td>Buy 1 S</td>
<td>-100</td>
<td>50</td>
<td>200</td>
</tr>
<tr>
<td>Borrow 40</td>
<td>40</td>
<td>-43.20</td>
<td>-43.20</td>
</tr>
<tr>
<td>Total</td>
<td>0</td>
<td>6.80</td>
<td>6.80</td>
</tr>
</tbody>
</table>

An important concept for option pricing is the “hedge ratio,” or delta.

The hedge ratio is defined as the number of shares of stock held for each call option written in the risk-free portfolio.

The hedge ratio for this example is 1/2.

We’ll talk more about deltas later.
Option Valuation

- What if you assumed the price could only go up to 200 or down to 50 but you were wrong?
- In general, there are many more steps in a tree than just 2.
- By having more steps, we make the model much more realistic.

Option Valuation

- When we have a more complicated tree, we just solve the problem at each node of the tree starting at the end.
- We get a new hedge ratio at each node.
- The more nodes we add, the smaller the jump between each node needs to be.
- On Wall Street, analysts use computer programs with many nodes.

Option Valuation

- Numerical example:

  Stock Price Tree

  100  102  104

  98   100

  Let's see how to value a call option with $X = 100$ using this tree and $r_f = 2\%$.

Option Valuation

- Numerical example:

  Option Price Tree

  $C_h$

  $C_i$

  4

  0

  We need to solve for each of the $C$ values in this diagram. $C_i$ is particularly easy.

Option Valuation

- Numerical example:

  Option Price Tree

  $C_h$

  $C_i$

  4

  0

  To get $C_h$ we need to calculate a hedge ratio. The ratio is:

  $H = \frac{C_+ - C_-}{S_+ - S_-} = 1$
Now we know that we could write 1 call and hold 1 stock and the payoff in the last node (assuming $S = 102$) is riskless.

Since this is a half period, $r_f = 1\%$

So $(S - C) = PV(100)$.

$102 - C = 99$.

$C = 102 - 99 = 3$.

Back to the tree diagram.

Numerical example:

To get $C$ we need to find delta again.

$$H = \frac{C^+ - C^-}{S^+ - S^-} = .75$$

Now we know that we could write 1 call and hold .75 stock and the payoff in the first node is riskless.

Since this is a half period, $r_f = 1\%$

If we short 1 call and buy .75 shares:

- We get $.75(98) = $73.50 if $S$ goes down.
- We get $.75(102) - 3 = $73.5 if $S$ rises.

With .75 shares and 1 short call, we get $73.50$ no matter what.

So $.75S - C = PV(73.5)$.

$75 - C = 72.77$.

$C = 75 - 72.77 = 2.23$.

Back to the tree diagram.

You do not need to know how to construct trees exactly like this.

If you ever use a tree algorithm, you should know what it is doing.

The tree model uses dynamic hedging.

When we allow for infinitely many nodes, this model converges to the Black-Scholes option pricing model.
Remember this tree?

$S_0 \rightarrow S_\pm \rightarrow S_{++} \rightarrow S_{+++}$

How many paths are there to each node?

- $S_0$
- $S_+$
- $S_-$
- $S_{++}$
- $S_{+++}$
- $S_{++-}$
- $S_+-$
- $S_{--}$

With many nodes, the distribution of prices becomes lognormal.

Option Valuation

- The Black-Scholes formula:

$$C_0 = S_0 N(d_1) - X e^{-rT} N(d_2)$$

$$d_1 = \frac{ln(S_0/X) + (r + \sigma^2/2)T}{\sigma T^{1/2}}$$

$$d_2 = d_1 - \sigma T^{1/2}$$

- Components of the formula:
  - $C_0$ = call price, $S_0$ = stock price, $X$ = strike.
  - $N(d^*)$ = cumulative normal distribution.
  - $e = 2.71828...$, $ln$ = natural logarithm.
  - $r$ = risk-free rate, $T$ = maturity.
  - $\sigma$ = standard deviation of log returns.

- You can calculate this with a calculator as long as you know the normal distribution.

Option Valuation

- Using $C$, we get put prices with PC parity.
- The BS model makes many assumptions.
  - The stock pays no dividends.
  - $r$ and $\sigma$ are constant over the life of the call.
  - Stock prices are continuous - no jumps.
  - Constant dynamic hedging is possible.
- More complicated models drop these.
- Binomial trees usually work.

Option Valuation

- We can observe all the components of the BS formula except for $\sigma$.
- Often we will calculate the value of $\sigma$ implied by one option value and apply it to valuing other options.
- Volatilities calculated this way are called implied volatilities.
Option Valuation

- One way to adjust BS for dividends and possible early exercise is to calculate BS values assuming early exercise and exercise at maturity.
- After calculating both types of call values, we pick the maximum value.
- It’s probably better to use a tree model for possible early exercise.

We can get the hedge ratio, delta, quite easily from the BS formula.

- With BS, \( \Delta = N(d_1) \).
- We can interpret delta as \( \Delta C/\Delta S \).
- This means that delta is just the slope of the graph that we saw before.

Option Valuation

- The delta of a call is the number of shares we need to be hedged with 1 call written.
- The delta of a call is positive.
- The delta of a put is always negative.
- The delta of a put is \( \Delta P/\Delta S = N(d_1) - 1 \).
- The put’s delta is the percentage of our stock holdings we need to put in T-bills to create a synthetic protective put position.

Summary:

- Option values depend on lots of variables.
- We can place bounds on option values.
- Early exercise is never optimal for a call that does not pay dividends.
- If a call pays large dividends, early exercise may be optimal.
- Puts with \( D = 0 \) may require early exercise.

Summary:

- Binomial tree models are used to value call options.
- With lots of nodes, a binomial tree becomes the Black-Scholes model.
- The delta of a call is \( \Delta C/\Delta S = N(d_1) \).
- The delta of a put is \( \Delta P/\Delta S = N(d_1) - 1 \).