14 Pricing Other Securities

In this section of the notes we discuss pricing models for particular securities. We have already discussed the pricing of options. Now we turn our attention to bonds and forward, future, and swap contracts. As in most of the class, our purpose is to familiarize ourselves with the type of model that exists for these securities rather than to get to know each possible model. To price these securities, we will apply the no-arbitrage in continuous time results that we have discussed. The reading that accompanies this section of the notes is Duffie's chapter 7.

14.1 Forward Contracts

We will discuss the pricing of forward contracts first. A forward contract is easy to price because it is just a promise to transact in the future. For example, if I go long a forward contract on wheat, I promise to buy wheat at a specified time in the future for the price that we negotiate today. The variable of interest is the price that we negotiate today, usually called the forward price. When you engage in a forward contract, you put no money down today, so the true "price" of a forward contract when it is initiated is zero. You put no money down today, but you either win or lose money over time, depending on how the price of the asset underlying the contract evolves.

Suppose that we are negotiating the futures price of a claim of $W$ (for wheat) in the future. Thus, our forward contract pays $F(t) - W$ at maturity, where $F(t)$ is the price we negotiate today. The forward contract pays nothing before maturity. Using our no-arbitrage results, we know that the price of any security is its discounted expected value under risk-neutral probabilities. If we let the discount rate vary through time,
this implies
\[
0 = E_t^Q \left[ \exp \left( - \int_t^T r(s) \, ds \right) \, W - F(t) \right].
\] (296)

Solving for the forward price,
\[
F(t) = \frac{E_t^Q \left[ \exp \left( - \int_t^T r(s) \, ds \right) \, W \right]}{E_t^Q \left[ \exp \left( - \int_t^T r(s) \, ds \right) \right]}.
\] (297)

Assuming that there exists a zero-coupon bond that matures at time \( T \) with price
\[
\Lambda_{t,T} = E_t^Q \left[ - \int_t^T r(s) \, ds \right]
\]
the forward price reduces to
\[
F(t) = \frac{1}{\Lambda_{t,T}} E_t^Q \left[ \exp \left( - \int_t^T r(s) \, ds \right) \, W \right].
\] (298)

If \( W \) and \( r(s) \) are statistically independent with respect to measure \( Q \) then we can write \( F(t) = E_t^Q(W) \), which implies that the forward price is a martingale under \( Q \).

As a specific example, suppose that we want to price a forward on a security with a price \( S(t) \) and a dividend process (either positive or negative) \( D(t) \). We know that the price of this security must equal
\[
S(t) = E_t^Q \left[ \exp \left( - \int_t^T r(s) \, ds \right) \, S(T) + \int_t^T \exp \left( - \int_t^u r(u) \, du \right) \, dD(s) \right].
\] (299)

The final value of our contract will be \( F(t) - S(T) \). Assuming again that there exists a zero-coupon bond of the right maturity, the forward price should be
\[
F(t) = \frac{1}{\Lambda_{t,T}} E_t^Q \left[ \exp \left( - \int_t^T r(s) \, ds \right) \, S(T) \right],
\] (300)
which, using (299), is

\[
F(t) = \frac{1}{\Lambda_{t,T}} \left( S(t) - E_{t}^{Q} \left[ \int_{t}^{T} \exp \left( - \int_{t}^{s} r(u)\,du \right) dD(s) \right] \right). \tag{301}
\]

If the short rate (the interest rate) is deterministic, then

\[
F(t) = \frac{S(t)}{\Lambda_{t,T}} - E_{t}^{Q} \left[ \int_{t}^{T} \exp \left( \int_{t}^{T} r(u)\,du \right) dD(s) \right]. \tag{302}
\]

which is also known as the cost of carry formula or the spot-forward parity theorem.

When we explain forward pricing to the MBAs, we generally make several simplifying assumptions that make the formula tractable. We assume, in particular, that the risk-free rate is constant and that the dividend yield of the underlying asset, the total dividends paid during a year divided by the initial price, is constant. If we express the risk-free rate as an arithmetic return per year rather than a continuously-compounded return, the cost of carry formula is written,

\[
F(t) = S(t)(1 + r_{f} - d)^{(T-t)}, \tag{303}
\]

where \(d\) is the dividend yield of the asset. The dividend yield can be negative if you are required to store the underlying asset and you derive no benefit from storing it (like wheat). The spot-forward parity theorem can be derived from fairly simple arbitrage arguments (suppose the forward was overpriced - short the future, buy the spot, and borrow at \(r_{f}\)).

There are a few specializations of the spot-forward parity theorem for certain cases. One specialization of note is called the interest rate parity theorem, or covered interest arbitrage. It involves currency forward contracts and risk-free securities in two countries. If we express exchange rates in terms of dollars per foreign currency

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and the risk-free rates as arithmetic holding-period returns, then

\[ F(t) = \frac{1 + r_{US}(t)}{1 + r_{FOR}(t)} E(t), \quad (304) \]

where the maturity of the forward contract must match the maturity of the risk-free bonds in both countries. Again, as long as some simplifying assumptions are met, this relation can be derived from simple arbitrage arguments.

14.2 Futures and Swaps

Under the simple assumptions we made above, forward contracts are fairly easy to price. A related contract called a futures contract is also fairly easy to price. Futures contracts are just like forward contracts except that futures contracts require investors to post margin, or collateral, and they practice marking to market, the process of determining each investor’s gains or losses each trading day. If an investor loses enough money then he has to post more margin to ensure that he will honor his futures obligation. When we teach MBAs about futures and forwards, we usually just give them the cost of carry formula, (303), and tell them that it holds approximately for both types of contracts. However, marking to market introduces a subtle difference between futures and forward pricing. This difference is explored in a paper by Cox, Ingersoll and Ross.\(^{11}\)

Another related financial contract is the swap. A swap is an agreement to exchange the cash flows of two similar assets for some period of time. For example, it is common to swap the interest payments from a fixed-coupon bond for those of a floating-coupon bond. It is also common to swap payments in one currency for payments in another. Like futures and forwards, there is a swap price at which both parties to a swap are

willing to engage in the contract with no money down. For an interest rate swap, the price is an interest rate. For FX swaps, it is an exchange rate. A swap can be thought of as a portfolio of forward contracts and priced accordingly.

14.3 Bonds

There is a huge literature on bond pricing. We will only be able to touch on a couple of bond pricing ideas. There is a lot of institutional knowledge about bonds that you should get somewhere. If you don’t know what bonds are, you should let me know.

Like other assets, people think that bond returns should reflect bond risks. Early work on bond pricing developed a measure of bond risk that is still used by practitioners today that is called duration. A bond’s duration is equal to the derivative of the bond’s price with respect to the bond’s implied interest rate, or the bond’s yield to maturity. A second measure of a bond’s risk is its convexity, which is the second derivative of the bond’s price with respect to the bond’s yield. We will not discuss duration and convexity, but you should know these concepts for future reference. You can read about them in any investments textbook for MBAs.

What we are really interested in is what is called the term structure of interest rates. The term structure is the pattern of interest rates available on bonds of different maturities. For practitioners, the term structure is conveniently summarized by the yield curve, which is reported daily in the Wall Street Journal. An example of a recent yield curve appears below.

We are interested in the yield curve because it tells us how to discount cash flows that are coming in the future. If we can accurately forecast the yield curve, we can make profits by buying and selling bonds appropriately. Furthermore, many of the newer derivatives being traded depend somehow on the yield curve.
We will model the yield curve by specifying a stochastic process for the short rate, or the instantaneous interest rate. Using the same notation that we have above, the price of a zero-coupon bond satisfies

\[ \Lambda_{t,T} = E_t^Q \left[ \exp \left( \int_t^T -r(u) \, du \right) \right] \quad (305) \]

A coupon bond can be thought of a portfolio of zero-coupon bonds with different face values. If we can price any zero-coupon bond then we can price any coupon bond as well.

There are several different types of term structure models available. We will just look at one simple model that is a one-factor model. One-factor models assume that the entire term structure is just a function of the short rate. Multi-factor models can assume that the term structure is a function of the short rate and some long-term
rate, or the short rate and two or three other factors.

All one-factor bond pricing models assume a stochastic differential equation of the form

\[ dr(t) = \mu(r,t)dt + \sigma(r,t)dW(t). \]  \hspace{1cm} (306)

The model that we will derive as an example of a term structure model is due to Cox, Ingersoll and Ross (CIR).\(^{12}\) The CIR model assumes a particular mean-reverting process for the short rate,

\[ dr = A(\bar{x} - r)dt + \sigma \sqrt{r}dW. \]  \hspace{1cm} (307)

In another paper in the same issue of Econometrica, Cox, Ingersoll, and Ross motivate the parameters \(A, \bar{x},\) and \(\sigma\) in a general equilibrium model. This general equilibrium model is quite famous - it is continuous-time analog of the Lucas consumption-based model that we discussed before. We will not have time to cover this model. You may want to investigate it on your own.

To get bond prices for the CIR model, we need to convert the process that we have assumed for the short rate, (307), into the expectation under \(Q\) of the stochastic integral of the short rate from \(t\) to \(T\), (305). We can do this with a mathematical relation between partial differential equations (PDE) and stochastic differential equations (SDE) known as the Feynman-Kac relation. For any one-factor model of the term structure, the Feynman-Kac relation implies that

\[ \Lambda_{t,T} = f(r,t), \]  \hspace{1cm} (308)

where \( f \) satisfies the PDE and boundary condition,

\[
Df(x, t) - xf(x, t) = 0,
\]
\[
f(x, T) = 1,
\]

and where

\[
Df(x, t) = f_t(x, t) + f_x(x, t)\mu(x, t) + \frac{1}{2}f_{xx}(x, t)\sigma(x, t)^2.
\]

As usual, to solve the Feynman-Kac relation for bond prices, we need to guess a solution and then verify that it is correct. Let’s suppose that the solution to our problem is

\[
f(x, t) = H_1(T - t)e^{-H_2(T-t)x},
\]

where \( H_1 \) and \( H_2 \) are functions of time to maturity, \((T - t)\). Verifying that this is the right form takes a little bit of algebra. We start by computing the components of the Feynman-Kac relation. The boundary condition is easiest to look at,

\[
H_1(0)e^{-H_2(0)x} = 1.
\]

Since this condition must hold for any value of \( x \), we can infer that

\[
H_1(0) = 1, \quad H_2(0) = 0.
\]

Now we turn to the PDE term,

\[
f_t - H_1H_2e^{-H_2x}A(\bar{x} - x) + \frac{1}{2}H_1H_2^2e^{-H_2x}\sigma^2x = xH_1e^{-H_2x}.
\]
We can compute the derivative of $f$ with respect to $t$,

$$f_t = -H_te^{-H_2x} + H_1H_2xe^{-H_2x}, \quad (315)$$

where $H_t$ is the derivative of $H$, with respect to $t$. Substituting this expression into (314), we get

$$-H_te^{-H_2x} + H_1H_2xe^{-H_2x} - H_1H_2e^{-H_2x}A(x - x) + \frac{1}{2}H_1H_2^2e^{-H_2x}\sigma^2x = xH_1e^{-H_2x}, \quad (316)$$

and dividing both sides by $e^{-H_2x}$,

$$-H_t + H_1H_2x - H_1H_2A(x - x) + \frac{1}{2}H_1H_2^2\sigma^2x = xH_1. \quad (317)$$

In order to make this differential equation hold for all values of $x$, we set the sum of all constant terms to zero,

$$H_t = -H_1H_2Ax, \quad (318)$$

and we set the coefficients on $x$ equal to zero as well,

$$H_2 = -H_2A - \frac{1}{2}H_2^2\sigma^2 + 1. \quad (319)$$

We have two partial differential equations in (318) and (319) and two boundary conditions in (313) to solve. We solve these equations again by proposing a form for the solution and verifying that it works. For the second constant, let’s try the form,

$$H_2(t) = \frac{2(e^{\gamma t} - 1)}{(\gamma + A)(e^{\gamma t} - 1) + 2\gamma}, \quad (320)$$

where

$$\gamma = \sqrt{(A^2 + 2\sigma^2)}. \quad (321)$$
We can see right away that this form meets the boundary condition $H_2(0) = 0$. The derivative of particular form with respect to $t$ is,

$$H_{2t} = \frac{2\gamma e^{\gamma t}[(\gamma + A)(e^{\gamma t} - 1) + 2\gamma] - (\gamma + A)\gamma e^{\gamma t}2(e^{\gamma t} - 1)}{[(\gamma + A)(e^{\gamma t} - 1) + 2\gamma]^2}$$

$$= \frac{4\gamma^2 e^{\gamma t}}{[(\gamma + A)(e^{\gamma t} - 1) + 2\gamma]^2} \equiv \frac{4\gamma^2 e^{\gamma t}}{v^2} \quad (322)$$

where $v$ is just a shorter notation for the term $v \equiv [(\gamma + A)(e^{\gamma t} - 1) + 2\gamma]$. Our condition is

$$\frac{4\gamma^2 e^{\gamma t}}{v^2} = \frac{-2A(e^{\gamma t} - 1)}{v} + \frac{-2\sigma^2(e^{\gamma t} - 1)^2}{v^2} + 1 \quad (323)$$

or

$$4\gamma^2 e^{\gamma t} = -2A(e^{\gamma t} - 1)v - 2\sigma^2(e^{\gamma t} - 1)^2 + v^2. \quad (324)$$

Expanding the term $v^2$,

$$v^2 = (\gamma + A)^2(e^{\gamma t} - 1)^2 + 4\gamma(\gamma + A)(e^{\gamma t} - 1) + 4\gamma^2$$

$$= (\gamma + A)^2(e^{\gamma t} - 1)^2 + 4\gamma^2 e^{\gamma t} - 4\gamma A - 4\gamma^2 + 4\gamma A e^{\gamma t} + 4\gamma^2. \quad (325)$$

Combining and rearranging,

$$4\gamma^2 e^{\gamma t} = -\left[2A(\gamma + A) + 2\sigma^2 - (\gamma + A)^2\right](e^{\gamma t} - 1)^2 - 4\gamma A e^{\gamma t}$$

$$+ 4\gamma A + 4\gamma^2 e^{\gamma t} - 4\gamma A - 4\gamma^2 + 4\gamma A e^{\gamma t} + 4\gamma^2. \quad (326)$$

Cancelling out like terms and dividing through by $(e^{\gamma t} - 1)^2$,

$$2A(\gamma + A) + 2\sigma^2 - (\gamma + A)^2 = 0 \quad (327)$$
Solving this equation for $\gamma$ with the quadratic formula gives the solution

$$\gamma = \sqrt{(A^2 + 2\sigma^2)},$$

which is exactly how we defined $\gamma$ in (321), so we’re done verifying the form for the function $H_2$. We will leave verifying the form for $H_1$ for a homework exercise - the method for verifying it is the same.

So what have we done? We have shown that the price of any zero-coupon bond is a function of the parameters that specify the process for the short rate, $A$, $\bar{x}$, and $\sigma$, and the short rate itself. We have shown that under our assumed process for the short rate, (307),

$$\Lambda_{t,T} = H_1(T - t)e^{-H_2(T-t)r},$$

where

$$H_1(t) = \left[\frac{2\gamma e^{(\gamma + A)t/2}}{(\gamma + A)(e^{\gamma t} - 1) + 2\gamma}\right]^{2A^2/\sigma^2},$$

$$H_2(t) = \frac{2(e^{\gamma t} - 1)}{(\gamma + A)(e^{\gamma t} - 1) + 2\gamma},$$

$$\gamma = \sqrt{(A^2 + 2\sigma^2)}.$$  

Using these equations, we could draw the yield curve if we knew the parameter values and current value of the short rate. In other words, we could figure out how to discount properly a certain payment that came at any point in the future. We have basically solved for the term structure.

There are lots of other term structure models that are described briefly in Duffie. One particularly famous model is the Vasicek model, which assumes a slightly different form for the stochastic process for the short rate.
14.4 Homework Problems

1. Verify that the term $H_1$ in the CIR model has the form

$$H_1(t) = \left[ \frac{2\gamma e^{(\gamma + A)t/2}}{(\gamma + A)(e^{\gamma t} - 1) + 2\gamma} \right]^{2A\eta/\sigma^2}.$$ 

2. What if the annual arithmetic risk-free return was 5%, the spot price of an ounce of gold was $300, the forward price for gold 1 year from now was $325, and an ounce of gold would cost you $3.00 to store for one year. Can you make an arbitrage profit? How would you do it and how much money per ounce of gold would you make?

3. You will need to do this assignment on a computer. Plot the yield curve for zero-coupon bonds of maturities up to thirty years for a CIR world with parameter values $A = 0.2$, $x = 0.05$, $\sigma = 0.17$, and short rates of 0.03, 0.04, and 0.05.