Remarks on JL Theorem:

1. **Dimension reduction for** $x$:
   From dim. $n$ to dim. $k \approx \log |x|$ ($\ll n$ usually).

2. The conclusion of JL Theorem does not depend on $n$ (except for normalization).
   And it should not, since one can always put $x$ in $\text{span}(x)$, which has dimension $\leq N$. Thus JL Theorem holds for $x \in$ Hilbert space

2.5. The dimension reduction map $A$ is non-adaptive (does not depend on $x$).

4. JL Theorem gives a \textbf{embedding} of the metric space $x \in \ell^2$ into $\ell^k$ ($= (\mathbb{R}^k, \| \cdot \|_2)$). In fact, a nearly isometric embedding

   $X \subset \ell^k$, $k \approx \log |x|$

Thus: if a finite metric space $X$ can be embedded into a Euclidean space ($\ell^2$), then $X$ can be embedded nearly isometrically into a Euclidean space $\ell^k$ of dimension $k \approx \log N$.

**Note:** not all finite metric spaces can be embedded nearly isometrically (or even isomorphically with const. bounds) into a Euclidean space [Bourgain '85?].
5) JL Transform uses random projection as a means of dimension reduction. Other random linear operators are possible, too.

**HW**

Prove that JL Theorem holds for $A : \mathbb{R}^n \rightarrow \mathbb{R}^k$ given by $A = \frac{1}{\sqrt{k}} \mathbf{G}$, where $\mathbf{G}$ is a Gaussian random matrix (with i.i.d. entries $G_{ij} \sim N(0,1)$).

In fact, you may prove that this holds for a general sub-gaussian random matrix $\mathbf{G}$, with entries satisfying

$$
E[G_{ij}] = 0, \quad \text{Var}(G_{ij}) = 1, \quad \|G_{ij}\|_2 \leq K = \text{const}
$$

6) "Fast JL Transforms" $A$ are known (based on FFT) [Ailon-Chazelle].

7) JL Theorem is a result about dimension reduction in $l_2$-norm.

There is no dimension reduction in $l_1$-norm.

[Brinkman-Cherrier, Lee-Naor]

**Literature**

- J. Matousek, Lectures on discrete geometry, Section 15.2 - proof of JL
- Ailon, Chazelle - fast JL
- Achlioptas - JL for other maps $A$
2.6. More on concentration: a survey

- Recall that we obtained the concentration results (for both sphere and Gaussian space) as an immediate consequence of:
  (i) isoperimetric inequality;
  (ii) bound on the measure of the extreme sets.

In a similar way, one obtains concentration results for several other measure metric spaces:

1. **Discrete cube** \((\{0,1\}^n, d, \mathbb{P})\)

   - **d = Hamming distance**
   - **\(\mathbb{P} = \text{uniform}$$

   \[
   \mathbb{P} \left[ |f(x) - E_\mathbb{P}(f)| > \varepsilon \right] \leq 2 \exp \left( - \frac{c n \varepsilon^2}{\|f\|_\mathbb{P}^2} \right), \quad \varepsilon > 0
   \]

   (Note: \(x_i\) are iid Bernoulli (1/2))

2. **Symmetric group** (of all permutations of \(1, 2, \ldots, n\))

   \((S_n, d, \mathbb{P})\), where \(d = \text{Hamming dist}, \ \mathbb{P} = \text{uniform}$$

Then (*) holds.

3. **Expanders graphs**

4. **Riemannian manifolds with strictly positive curvature**

   (with \(\text{Ricci curvature} \geq c\text{onst.}$$
   
   [\text{M. Gromov}]$$

   This class includes \(S^n\) and also (4), (5), (6), (7), (8) below:

   (see below Thm 2.4)

5. **The set of orthogonal matrices** \((O(n), \|\cdot\|_F, \mathbb{P})\)

   - \(\|\cdot\|_F = \text{Frobenius norm (a.k.a. Hilbert-Schmidt norm)}$$
   
   \[
   \|A\|_F = \sqrt{\sum_{i,j} A_{ij}^2}
   \]

   - \(\mathbb{P} = \text{uniform (Haar) measure}$$

Then (*) holds.

5. **Same for** \(SO(n)\); \(\text{det } U = 1\)
5) Grassmannians \( (\mathcal{G}_n,k, d, P) \)

\[ \mathcal{G}_n,k = \{ \text{k-dimensional subspaces in } \mathbb{R}^n \} \]

\[ d = \text{Hausdorff dist} \]

\[ d(E,F) = \sup_{x \in S^{n-1}} \text{dist} (x, S^{n-1} \cap E) \]

\[ P = \text{uniform (Haar) measure} \]

Then (\( ν \)) holds.

7) Stiefel manifolds \( (W_n,k, \| \cdot \|_F, P) \)

\[ W_n,k = \{ \text{n x k matrices with orthogonal columns} \} \]

Then (\( ν \)) holds.

\[ W_n,k \cong O(n)/O(n-k) \Rightarrow (7) \text{ can be deduced from (4).} \]

\[ S^{n-1} = W_{n,1}, \; SO(n) = W_{n,n-1} \Rightarrow \text{sphere and (5) can be deduced as well.} \]

\[ G_n,k = O_n/(O_k \times O_{n-k}) \Rightarrow (6) \text{ can also be deduced.} \]

- There are methods to prove concentration, which do not use isoperimetric inequalities:
  - semigroup tools
  - spectrum of the Laplacian (spectral gap) – Poincaré
  - log-Sobolev & entropy methods
  - transportation cost inequalities
  - martingale methods

Using these methods, one obtains concentration for several examples above (including sphere and Gauss space, and also:

8) Euclidean ball \( (B(0,1), \| \cdot \|_2, P) \)

[Ledoux Prop 2.7] – deduced from Gauss space.

9) Continuous cube \( ([0,1]^n, \| \cdot \|_2, P) \)

\[ \| f(x) - EF(x) \|_2 \leq C \| f \|_{lip} \]

[Ledoux Prop 2.8] – deduced from Gauss space

Warning: but not \( A \) convex body (e.g. \( B_1^n = \text{ball (} \ell_1^n \text{)} \))

(However see Borell’s inequality for \( \| \cdot \|_{L_2} \).

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Random vectors with density \( f(x) = e^{-U(x)} \), \( \text{Hess}(U) \geq c \cdot \text{Id} \) if curvature \( \geq c \)

\[ \Rightarrow (**) \text{holds} \]

[Ledoux Prop. 2.12] - semi-group tools; see also Thm. 2.15

Example: Gaussian r.v.; \( f(x) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{\|x\|^2}{2}} = \exp\left(-\frac{\|x\|^2}{2} - c\right) \)

Random vectors with independent bounded coard's \[ \text{[Talagrand]} \]

\( X = (X_1, \ldots, X_n), \ X_i \text{ independent, } 1 \leq i \leq 1 \ a.s. \)

Then
\[ \| f(x) - E f(x) \|_{L^2} \leq C \| f \|_{Lip} \]

for every convex function \( F : \mathbb{R}^n \to \mathbb{R} \) (e.g. for any norm on \( \mathbb{R}^n \))

[see Ledoux Cor. 4.10]

Literature:
- M. Ledoux, The concentration of measure phenomenon
  - comprehensive book
- V. Milman, G. Schechtman, Asymptotic theory
  - Chapters I. 2, I. 6, V.
- M. Talagrand, A new look at independence
  - tutorial paper
- M. Ledoux, M. Talagrand, Probability in Banach spaces
  - book.