Remarks

1. An band of width $\approx \frac{1}{\sqrt{n}}$ has constant measure:
   \[ \sigma_{n-1}(A_\varepsilon) = \frac{1}{2} \quad \text{for } \varepsilon \approx \frac{1}{\sqrt{n}}. \]

2. This happens for every equatorial band.

3. An equivalent form of Prop.: for a hemisphere $H$, its $\varepsilon$-neighborhood $A_\varepsilon$ has large measure:
   \[ \sigma_{n-1}(H \setminus \varepsilon) \geq 1 - \exp(-n \varepsilon^2/2) \quad (\star) \]
   (since $(H \setminus \varepsilon) = C_\varepsilon$

In fact, (3) happens for any subset of measure $\geq \frac{1}{2}$!

This follows from:

THM (Isoperimetric inequality for $S^{n-1}$)
Among all measurable sets $A \subseteq S^n$ with a given measure, and for given $\varepsilon$ spherical caps minimize the measure of $A_\varepsilon = \{ x \in S^n : \text{dist}(x, A) \leq \varepsilon \}$

THM (Concentration on $S^{n-1}$). Every measurable set $A \subseteq S^n$ with $\sigma_{n-1}(A) \geq \frac{1}{2}$ satisfies
   \[ \sigma_{n-1}(A_\varepsilon) \geq 1 - \exp(-n \varepsilon^2/2), \quad \varepsilon \geq 0 \]

Proof

\[ \sigma_{n-1}(A_\varepsilon) \geq \sigma(H_\varepsilon) \quad \text{(by the isoperimetric inequality)} \]
\[ \geq 1 - \exp(-n \varepsilon^2/2) \quad \text{(by } (\star)) \]

Q.E.D.
Remark (on the "isoperimetric inequality", the first theorem on p. 24).

The area of the boundary of $A$ is

\[ \sigma_{n-1}(A) = \lim_{\varepsilon \to 0} \frac{\sigma(A_\varepsilon) - \sigma(A)}{\varepsilon} \]

\[ = \frac{\sigma(\text{Cap}) - \sigma(\text{Cap})}{\varepsilon} \quad \text{(by Thm.)} \]

where $\text{Cap}$ is a spherical cap with $\sigma(A) = \sigma(\text{Cap})$.

\[ \Rightarrow \quad \sigma_{n-1}(A) = \sigma_{n-1} (\text{Cap}) \]

5) Cor (Classical Isoperimetric inequality for $S^{n-1}$)

Among all subsets of $S^{n-1}$ with a given area, spherical caps minimize the area of the boundary.

- Recall that this also holds in $\mathbb{R}^n$ - among all sets of a given volume, Euclidean balls minimize the surface area.

Literature: R. Vershynin, Lectures in geometric functional analysis

M. Ledoux, Concentration of measure phenomenon

- Comprehensive book

K. Ball, An elementary introduction into modern convex geometry

- Short and very readable
2.2 Concentration for Lipschitz functions

Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces.

A map \(f: X \rightarrow Y\) is called \textit{Lipschitz} if \(\exists L:\)

\[
d_Y(f(x_i), f(x_j)) \leq L \cdot d_X(x_i, x_j)
\]

for all \(x_i, x_j \in X\).

The smallest constant \(L\) is called the \textit{Lipschitz norm} of \(f\),

as is denoted \(\|f\|_{\text{Lip}}\).

\[\begin{array}{c}
\text{Examples} \\
1) \text{every differentiable function } f: \mathbb{R}^n \rightarrow \mathbb{R} \text{ is Lipschitz, } \|f\|_{\text{Lip}} \leq \|df\|_{\infty}, \text{ but not vice versa: } f(x) = |x| \text{ is Lipschitz but not differentiable at } 0. \\
\end{array}\]

\[\begin{array}{c}
\text{Lipschitz} \\
\text{Not Lipschitz}
\end{array}\]

2) Let \(\|\cdot\|\) be a norm on \(\mathbb{R}^n\).

Consider the function \(f: (\mathbb{R}, \|\cdot\|) \rightarrow \mathbb{R}\) given by

\[f(x) = \|x\|\]

Prove that the Lipschitz constant of \(f\) is the smallest \(L\) s.t.

\[\|x\| \leq L \|x\|_2 \quad \text{for all } x \in \mathbb{R}^n.
\]

Consequently, any norm \(\|\cdot\|\) on \(\mathbb{R}^n\) is a Lipschitz function

(this follows from the Open Mapping Theorem),

however, the Lipschitz constant may depend on the choice of

the norm (and on \(n\)).
Consider a Lipschitz function \( f: S^{n-1} \to \mathbb{R} \), \( \|f\|_{Lip} = L \).

Our goal is to show that the values of \( f \) are concentrated around one value \( M \).

Let \( A := \{ x : f(x) \leq M \} \) and \( \sigma_{n-1} (A) \geq \frac{1}{2} \).

By Concentration of Measure Theorem (p. 24),

\[
\sigma_{n-1} (A_{\varepsilon}) \geq 1 - \exp \left( -n\varepsilon^2 / 2 \right), \quad \varepsilon > 0.
\]

For every \( x \in A_{\varepsilon} \), \( \exists y \in A: d(x, y) \leq \varepsilon \)

\( \Rightarrow f(x) \leq f(y) + L\varepsilon \) (by def. of Lipschitz)

\( \leq M + L\varepsilon \) (since \( y \in A \)).

Thus, \( A_{\varepsilon} \subseteq \{ x : f(x) \leq M + \|f\|_{Lip} \varepsilon \} \).

But \( \sigma_{n-1} (A_{\varepsilon}) \) is large by \((\dagger)\) \( \Rightarrow \)

\( \sigma_{n-1} (\{ x : f(x) \leq M + L\varepsilon \}) \geq 1 - \exp \left( -n\varepsilon^2 / 2 \right). \)

Similar argument for \( B = \{ x : f(x) \geq M \} \) (do it!)

\( \sigma_{n-1} (\{ x : f(x) \geq M - L\varepsilon \}) \geq 1 - \exp \left( -n\varepsilon^2 / 2 \right). \)

Continuing \( \Rightarrow \)

\( \sigma_{n-1} (\{ x : |f(x) - M| > L\varepsilon \}) \leq 2\exp \left( -n\varepsilon^2 / 2 \right). \)

\( (\varepsilon \to \varepsilon / 2) \)
We have proved:

Let $f: S^n \to \mathbb{R}$ be a Lipschitz function, $L = \|f\|_{Lip}$, $M = \text{Med} |f|$.

$$\forall_{n-1} \left( x : |f(x) - M| > \varepsilon \right) \leq 2 \exp \left( - \frac{n \varepsilon^2}{2 L^2} \right), \quad \varepsilon > 0$$

"A Lipschitz function is almost constant on almost the entire sphere”.

2.3. Concentration in Gauss space.

$(S^n, \mathcal{d}, \nu_{n-1})$ can be replaced by $(\mathbb{R}^n, \|\cdot\|_2, \gamma_n)$

- geodesic dist

The argument consists of the same two major steps:

1. Isoperimetric inequality

Thus [Sudakov-Tsirelson, Borell '74-75]

Among all measurable sets $A \subset \mathbb{R}^n$ with a given measure $\gamma_n(A)$, and for given $\varepsilon$, half-spaces minimize $\gamma_n(A \varepsilon)$

2. Measure of the extreme sets

If $H$ is a half-space with $\gamma_n(H) \geq \frac{1}{2}$

then (by rotation invariance)

$$\gamma_n(H_{\varepsilon}) = \gamma_n \left\{ x \in \mathbb{R}^n : x_1 \leq \varepsilon \right\} = P \left\{ g_1 \leq \varepsilon \right\} \sim \mathcal{N}(0,1)$$

$$\geq 1 - \exp (-\varepsilon^2/2) \quad (\text{see p. 2})$$
Combining \( \odot \) & \( \ominus \) as in the spherical case:

**Thm.** (Concentration in the Gaussian space).

Every measurable set \( A \subseteq \mathbb{R}^n \) with \( \gamma_n(A) = \frac{1}{2} \) satisfies

\[
\gamma_n(A^c) \geq 1 - \exp\left(-\frac{t^2}{2}\right), \quad t > 0
\]

**Thm.** (Functional form)

Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a Lipschitz function, \( L = \| f \|_{Lip}, \quad m = \text{Med}(f) \).

Then

\[
\gamma_n(\lambda : |f(\lambda) - M| > \varepsilon) \leq 2\exp\left(-\frac{t^2}{2L^2}\right), \quad t > 0
\]

HW: Prove these two theorems (follow the argument for the sphere).

Probabilistic interpretation: \( g \sim N(0, I_n) \Rightarrow g = (g_1, \ldots, g_n), \quad g_i \sim N(0, 1) \text{iid} \).

**Thm.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a Lipschitz function, \( L = \| f \|_{Lip}, \quad m = \text{Med}(f) \).

Then, for \( g \sim N(0, I_n) \) one has

\[
P\left\{ |f(g) - M| > \varepsilon \right\} \leq 2\exp\left(-\frac{t^2}{2L^2}\right), \quad t > 0
\]

Literature - see p. 25