6.7. Matrices satisfying R.I.P.

Goal: construct matrices which satisfy RIP with small number of measurements:
\( \alpha, \beta = \text{const (e.g. } \frac{1}{2}, 2) \),
\( n \) arbitrary, \( k \) arbitrary,
\( m \leq k \log n \) (or \( k \log^\alpha n \))

Open question: deterministic constructions

**Theorem (sub-gaussian RIP).** Let \( A \) be an \( m \times n \) matrix whose rows are independent, isotropic sub-gaussian vectors.

Then \( \overline{A} = \frac{1}{\sqrt{m}} A \) satisfies the following \( \forall k \leq n, \sigma \in (0, 1) \):

if \( m \geq C \delta^2 k \log(en/k) \)

then \( \overline{A} \) satisfies RIP with \( \alpha = 1 - \delta, \beta = 1 + \delta, k \)

with prob. \( \geq 1 - 2 \exp(-Ct^2) \).

**Proof:**

- Fix \( J \subset [n], |J| = k \).

Apply Theorem for \( A_J \) (\( m \times k \) matrix; independent, isotropic sub-gaussian rows)

\[
\gamma_m - \delta \sqrt{-t} \leq \sigma_{\min}(A_J) \leq \sigma_{\max}(A_J) \leq \gamma_m + \delta \sqrt{t} + t \quad (\#)
\]

with prob. \( \geq 1 - 2 \exp(-Ct^2) \).

- Union bound: \((\#)\) holds simultaneously for all \( J \) with prob.

\[
1 - 2 \binom{n}{k} \exp(-Ct^2) \geq 1 - 2 \exp(k \log \frac{en}{k} - Ct^2)
\]

\[
\geq 1 - 2 \exp\left(k \log \frac{en}{k}\right)
\]

if we choose \( t \sim \sqrt{k \log \frac{en}{k}} \).

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If \( m \geq C \delta^2 k \log \left( \frac{e^n}{n} \right) \) then (**) implies

\[
(1-\delta)S_m \leq S_{\min}(A) \leq S_{\max}(A) \leq (1+\delta)S_m \\
\Rightarrow \quad 1-\delta \leq S_{\min}(A) \leq S_{\max}(A) \leq 1+\delta \\
RIP \text{ with } \alpha = 1-\delta, \beta = 1+\delta, \ \kappa
\]

**DEQ.**

Remark. Similar result for matrices with independent subgaussian columns.

We proved: a subgaussian matrix \( A \) satisfies RIP with \( m \approx k \log(\frac{n}{k}) \).

Combining with Theorem p. 134, a \( k \)-sparse signal \( x \in \mathbb{R}^n \) can be exactly recovered from \( m \approx k \log \left( \frac{n}{k} \right) \) linear measurements given as \( y = Ax \).

Here \( A \) is an \( m \times n \) matrix with independent subgaussian rows.

Uniformity: One matrix \( A \) works for signals \( x \), with prob. \( 1 - \exp(-c m) \).

**Heavy-tailed RIP.**

Goal: random matrices with general independent isotropic rows (not subgaussian) satisfy RIP with

\[ m \approx k \log n \]

Motivation: \( A = \) random sample of \( m \) rows of \( \sqrt{m} U \) where \( U \) is a given orthogonal matrix

\[
\begin{align*}
U & \rightarrow_{m}^n \mathbb{A} \\
& \text{Clearly, rows of } \mathbb{A} \text{ are independent, isotropic.}
\end{align*}
\]
Example: \[ U = \text{DFT matrix}, \quad U_{ij} = \frac{1}{\sqrt{n}} \exp \left( -\frac{2\pi i j}{n} \right) \quad i,j \in \{0, \ldots, n-1\} \]

\[ \begin{aligned}
&\mathbb{R}^n \to \text{DFT} \quad \hat{f} = UF \\
&U \text{ orthogonal} \quad \Leftrightarrow \text{Parseval's identity} \quad \|\hat{f}\|_2 = \|f\|_2
\end{aligned} \]

Measurements \( y = Ax \equiv m \text{ random frequencies of } x \)

(or, if \( x \) is in Fourier space, \( m \) time-samples).

MR1.

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\[ \text{Thm (Heavy-tailed RIP): } \text{Let } A \text{ be an } m \times n \text{ matrix whose rows are independent, isotropic } \mathbb{R} \text{ vectors, and } |A_{ij}| = O(1) \text{ a.s.} \]

Then \( \bar{A} = \frac{1}{\sqrt{m}} A \) satisfies the following: \( \forall k \leq n, \quad \delta \in (0,1) \):

\[ \text{if } m \geq C \delta^{-2} k \log n \]

then \( \bar{A} \) satisfies RIP with \( \alpha = 1-\delta, \beta = 1+\delta, k \)

Open problem: \( m = k \log n ? \)

\[ \text{Difficulty: } \text{for a fixed } J, \quad \left| H_J \right| = k \]

we know RIP from Rudelson's theorem, but with polynomial prob.

\[ \Rightarrow \text{can't apply union but our exponential) } \]

Remark on \(|A_{ij}| = O(1)| \): Then false e.g. for rows \( \text{uniform } (\sqrt{n} \text{ reals}) \)

\[ \text{Indeed, } A \text{ would have } \geq n - m \text{ empty columns } \Rightarrow \text{not RIP} \]

Remark on \( \text{uncertainty principle} \).

Literature: [V, Introduction to how asymptote...]

[V, 9-hour course... see "slides" link in my webpage].