Apply Slepian's inequality:

\[
E \max_{(A)} = E \max_{(u,v) \in T} X_{u,v} \leq E \max_{(u,v) \in T} Y_{u,v}
\]

\[
= E \sup_{u \in S^{n-1}} \langle g, u \rangle + E \sup_{v \in S^{n-1}} \langle h, v \rangle
\]

\[
= E \|g\|_2 + E \|h\|_2
\]

\[
\leq \left( E \|g\|_2^2 \right)^{1/2} + \left( E \|h\|_2^2 \right)^{1/2}
\]

\[
= \sqrt{n} + \sqrt{n}.
\]

QED

Remark: For \( S_{\min}(A) = \min_{u \in S^{n-1}} \| A u \|_2 = \min_{(u,v) \in T} \max_{v \in S^{n-1}} \langle A u, v \rangle \),

a similar proof works, but Slepian's inequality

is replaced by a more general Gordon's inequality.

Lemma (Gordon's inequality) Consider two Gaussian processes

\((X_{u,v})_{u \in U, v \in V}\) and \((Y_{u,v})_{u \in U, v \in V}\).

Assume that the increments satisfy

\[
E |X_{u,v} - X_{u',v'}|^2 \leq E |Y_{u,v} - Y_{u',v'}|^2 \quad \forall u \in U; \ v, v' \in V;
\]

\[
E |X_{u,v} - X_{u',v'}|^2 \geq E |Y_{u,v} - Y_{u',v'}|^2 \quad \forall u \neq u'; \ u, u' \in U; \ v, v' \in V.
\]

Then

\[
E \inf_{u \in U} \sup_{v \in V} X_{u,v} \leq E \inf_{u \in U} \sup_{v \in V} Y_{u,v}.
\]
• Gordon's Thm is about the expectation of $\text{Smin}(A), \text{Smax}(A)$

"High probability" version follows from concentration:

• Recall Gaussian Concentration (Cor. p. 30):

$$\|f(g) - Ef(g)\|_2 \leq C \|f\|_\infty \quad \forall f: \mathbb{R}^n \to \mathbb{R} \text{ Lipschitz}$$

In particular,

$$P\{\|f(g) - Ef(g)\| > t\} \leq 2\exp\left(-c t^2/\|f\|_\infty^2\right), \quad t > 0 \quad (\ast)$$

• let $A$ be as in Gordon's Thm, i.e. $N \times n$ matrix with iid $N(0,1)$ entries

View $A \in \mathbb{R}^{N \times n}$ as a Gaussian vector: $A \sim N(0, \mathbb{I}_n)$

$\text{Smax}(A) = \|A\| \leq \|A\|_2 = \|A\|_F \quad \Rightarrow \quad A \mapsto \text{Smax}(A)$ is 1-Lipschitz $\mathbb{R}^{N \times n} \to \mathbb{R}$

Similarly for $\text{Smin}$ (and for all intermediate singular values - check!)

• Thus, combining Gordon's theorem with Gaussian concentration $(\ast) \Rightarrow$

Cor: let $A$ be a finite random matrix, $N \times n$ (entries $\sim N(0,1)$ iid)

Then for every $t \geq 0$, with probability $\geq 1 - 2\exp(-t^2)$,

$$\sqrt{N} - \sqrt{n} - t \leq S_{\text{min}}(A) \leq S_{\text{max}}(A) \leq \sqrt{N} + \sqrt{n} + t.$$}

See [G. Aubrun] for a more precise statement for Wigner matrices.

Problem: Can Gordon's Theorem be generalized (to general Ginibre, not necessarily Gaussian)?
Non-asymptotic result in this direction:

Thus \((\text{Bai-Yin Law})\):

Let \(A = A_{n,n}\) be a random matrix whose entries are iid, mean zero, variance 1, fourth moment \(\infty\). Assume \(N, n \rightarrow \infty, n/N \rightarrow \lambda \in [0,1]\).

Then with probability 1,

\[
\begin{align*}
s_{\min}(A) &= \sqrt{N} - \sqrt{n} + o(\sqrt{n}), \\
s_{\max}(A) &= \sqrt{N} + \sqrt{n} + o(\sqrt{n}).
\end{align*}
\]

Remarks:

1. Equivalently, in M-P law, the largest and smallest eigenvalues of Wishart matrix \((W = \frac{1}{n} A A^T)\) converge to the endpoints of M-P law.

(\(\exists \) outlier in M-P Law)

2. It is essential that 4\textsuperscript{th} moment be finite.

If \(E|A_{ij}|^4 = \infty\) then with high prob. \(\exists\) an entry of \(A\) whose magnitude is \(\gg \sqrt{n}\).

\(\Rightarrow\) \(s_{\max}(A) \geq |\text{that entry}| \gg \sqrt{n}\).

3. A non-asymptotic version of Bai-Yin law see in \([O. Feldheim, A. Sodin, 2011]\).

The asymptotic distribution of both \(s_{\max}(A), s_{\min}(A)\)

is Tracy-Widom law.
Extreme eigenvalues and near isometries:

For a tall matrix $A$ ($N \gg n$), both $\sigma_{\min}(A) = \sigma_{\max}(A) = \sqrt{N}$

(=) all $\sigma_j(A) = \sqrt{N}$

(=) all $\sigma_j(\frac{1}{\sqrt{N}} A) = 1$

(=) $\frac{1}{\sqrt{N}}$ is a "near isometry" (i.e. $\frac{1}{\sqrt{N}} A^* A = I$) (ie) $\frac{1}{\sqrt{N}} \|Ax\|_2 = \|x\|_2$ \forall $x \in \mathbb{R}^n$

Rigorously:

Lemmas (Near isometries) Consider a matrix $B$ s.t. for some $\delta > 0$,

$\|B^* B - I\| \leq \max(\delta, \delta^2)$ \hspace{1cm} (4)

Then

$1 - \delta \leq \sigma_{\min}(B) \leq \sigma_{\max}(B) \leq 1 + \delta$ \hspace{1cm} (4')

Proof

(4) $\Leftrightarrow$ $\|B^* B - I\| \leq \max(\delta, \delta^2)$ $\forall x \neq 0$.

(4') $\Leftrightarrow$ $\|Bx\|_2 - 1 \leq \delta$ (due to the geometric meaning of extreme singular values $p, q$).

Then (4) $\Leftrightarrow$ (4').

Literature: [V, Introduction to new asymptotics...]

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