(a) \[ \| P_U(x) \| = \left\| \sum_{n=1}^{\infty} \langle u_n, x \rangle u_n \right\|^2 = \sum_{n=1}^{\infty} \| \langle u_n, x \rangle u_n \|^2 \]

by the orthogonality (Lemma 6.23)

= \sum_{n=1}^{\infty} |\langle u_n, x \rangle|^2 by the orthonormality (\| u_n \| = 1)

\leq \| x \|^2 by Bessel's inequality.

Hence \[ \| P_U(x) \| \leq \| x \| \text{ for all } x \in H, \] (\(*\))

Thus \( P \) is bounded.

\[ \| P_U \| = \sup_{\| x \| = 1} \| P_U(x) \| \]

\( P_U \) is linear because

\[ P_U(ax + by) = \sum_{n=1}^{\infty} \langle u_n, ax + by \rangle u_n \]

= \sum_{n=1}^{\infty} \left( a \langle u_n, x \rangle u_n + b \langle u_n, y \rangle u_n \right) by the linearity of the inner product

= a \sum_{n=1}^{\infty} \langle u_n, x \rangle u_n + b \sum_{n=1}^{\infty} \langle u_n, y \rangle u_n

= a P_U(x) + b P_U(y).

(b) From (\(*\)) in part (a), we know that \( \| P_U \| \leq 1 \).

Also, \( P_U(u_n) = \langle u_n, u_n \rangle u_n \) by the orthogonality

\[ = u_n \]

hence \( \| P_U(u_n) \| = \| u_n \| = 1 \), thus \( \| P_U \| = 1 \).
(c) \[ P_U^2 x = P_U \left( \sum_{n=1}^{\infty} \langle u_n, x \rangle u_n \right) \]
\[ = \sum_{n=1}^{\infty} \langle u_n, x \rangle P_U(u_n) \quad \text{by the linearity and the continuity of } P_U \]
\[ = \sum_{n=1}^{\infty} \langle u_n, x \rangle u_n \quad \text{by part (b) } \left( P_U(u_n) = u_n \right) \]
\[ = P_U x. \]
Hence \[ P_U^2 = P_U \]

(d) When \( U \) is an orthonormal basis, i.e. when \( U \) is complete.

This happens if

Indeed, \( (P_U x - x, x) \) is nothing else than the identity
\[ x = \sum_{n=1}^{\infty} \langle u_n, x \rangle u_n \quad \forall x \in \mathcal{H} \]
which is equivalent to the completeness of \( U \) by Theorem 6.26.

(2)(a) Let \( \mathbb{1}_{[a,b]} \) denote the function that takes value 1 on \([a,b]\) and zero elsewhere. Then
\[ \sum_{n=1}^{\infty} \left| \int_a^b u_n(x) \, dx \right|^2 = \sum_{n=1}^{\infty} \left| \int_0^1 \mathbb{1}_{[a,b]}(x) \overline{u_n(x)} \, dx \right|^2 \]
\[ = \sum_{n=1}^{\infty} \left| \langle u_n, \mathbb{1}_{[a,b]} \rangle \right|^2 \quad \text{by Parseval's equality} \]
\[ = \| \mathbb{1}_{[a,b]} \|_{L^2[0,1]}^2 \]
\[ = \int_a^b \mathbb{1}_{[a,b]}(x) \, dx = \int_a^b dx = b - a. \]
First solution: Assume $\bar{u}(u_n)$ is not complete.

Then $U^\perp$ is not empty, hence $\exists \, v \in U^\perp: \langle u_n, v \rangle = 0$,

Thus $U \cup \{v\}$ is an orthonormal set.

By Bessel's inequality,

$$\sum_n |\langle u_n, f \rangle|^2 + |\langle v, f \rangle|^2 = \|f\|^2 \quad \forall \, f \in L^2([0,1])$$

In light of part (a),

The condition (iv) in Problem 2 can be stated as

$$\sum_n |\langle u_n, f \rangle|^2 = \|f\|^2 \quad \text{for } f = 1_{[a,b]} \quad (A \, 0 < a < b < 1)$$

These two identities imply $|\langle v, f \rangle|^2 = 0$ for $f = 1_{[a,b]}$, hence

$$\langle v, f \rangle = 0 \quad \text{for } f = 1_{[a,b]}, \quad \forall \, 0 < a < b < 1.$$ 

By the linearity, summing up these for different $a, b$ yields

$$\langle v, f \rangle = 0 \quad \forall \text{ piecewise constant function } f$$

In particular,

$$\langle v, h_n \rangle = 0 \quad \forall \, n, \text{ where } (h_n) \text{ is the Haar wavelet basis}$$

(because the elements of that basis are piecewise-constant).

Thus

$$V = \sum \underbrace{\langle h_n, v \rangle}_{\|v\|} V_n = 0$$

Contradiction to $\|v\| = 1$. Q.E.D.
In light of part (a), we can write condition (x) as
\[ \sum_{n} |k_{n, f}|^2 = \| f \|^2 \quad \forall f = \Delta [a, b], \quad 0 < a < b < 1. \]
Since the left hand side equals \( \| P_{\Omega} (f) \|^2 \), we have
\[ \| P_{\Omega} (f) \|^2 = \| f \|^2. \]

Since \( f = P_{\Omega} (f) + (f - P_{\Omega} (f)) \), we have \( \| f \|^2 = \| P_{\Omega} (f) \|^2 + \| f - P_{\Omega} (f) \|^2 \)
by orthogonality,

hence
\[ \| f - P_{\Omega} (f) \|^2 = 0. \]

Thus
\[ (*) \quad (\text{id} - P_{\Omega}) f = 0 \quad \text{for } f = \Delta [a, b], \quad 0 < a < b < 1 \]
where id is the identity operator on \( L^2 [0, 1] \).

By the linearity of the operator \( (\text{id} - P_{\Omega}) \), we can sum \( (*) \) with different \( a, b \)'s so as to get
\[ (**) \quad (\text{id} - P_{\Omega}) f = 0 \quad \text{for all piecewise-constant } f. \]

Since the set of piecewise-constant functions is dense in \( L^2 [0, 1] \), any function \( g \in L^2 [0, 1] \) is the limit of some sequence of piecewise-constant functions \( f_k \in L^2 [0, 1] \). Hence the \( \Theta \)-continuity of the operator \( (\text{id} - P_{\Omega}) \) (Problem 1a) implies and \( (**) \) implies
\[ (\text{id} - P_{\Omega}) g = 0 \quad \forall g \in L^2 [0, 1]. \]
That is, \( P_{\Omega} g = g \quad \forall g \in L^2 [0, 1] \)
which means that \( \Omega \) is an orthonormal basis
(by the definition of \( P_{\Omega} \)).
(a) Since \( u(\cdot, t) \in L^2(\mathbb{R}) \), we can write its Fourier series \( u(t) \) as

\[
 u(x, t) = \sum_{n=-\infty}^{\infty} u_n(t) e^{inx}
\]

By the PDE,

\[
 u_t = u_{xxx}
\]

\[
 \sum_{n=-\infty}^{\infty} u_n'(t) e^{inx} = \sum_{n=-\infty}^{\infty} (in)^3 u_n(t) e^{inx}, \quad \text{thus}
\]

\[
 u_n'(t) = -in^3 u_n(t).
\]

Solving the O.D.E. \( \dot{z} = -in^3 z \) we have \( z = ce^{-in^3 t} \), hence \( u_n(t) = c_n e^{-in^3 t} \) for some constants \( c_n \).

Hence \( u(x, t) = \sum_{n=-\infty}^{\infty} c_n e^{-in^3 t} e^{inx} \).

Using the initial condition, \( f(x) = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx} \),

\[
 \hat{f}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx
\]

we find that \( c_n = \hat{f}_n \), thus the solution is

\[
 u(x, t) = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{i(x-n^3 t)}
\]

(\text{The part on } g^t \text{ was not graded}).

(b) \( u(\cdot, t) \in C^4(\mathbb{R}) \) if \( u(\cdot, t) \in H^{\frac{3}{2}+\varepsilon}(\mathbb{R}) \) for some \( \varepsilon > 0 \) which happens if

\[
 \sum_{n=-\infty}^{\infty} n^{3+\varepsilon} |\hat{f}_n|^2 < \infty \quad \text{for some } \varepsilon > 0.
\]