Math 214, Section 002: Midterm Exam 1
Instructor: Jenna Rajchgot
October 8, 2014, 1:10pm-2:00pm

Instructions

This exam consists of 1 cover page and 6 question sheets. Ensure that your exam has all 7 pages before you begin.

You are allowed to have one 3" × 5" note card.

Put your name on every page of the exam as it’s easy for pages to become separated.

Show your work and be sure to clearly indicate any assumptions that you are making. A correct answer under (minor) additional assumptions will receive partial credit.

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Do not cheat. If you cheat, you risk failing the course.

Name: Solutions

<table>
<thead>
<tr>
<th>Question</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Points:</td>
<td>30</td>
<td>20</td>
<td>30</td>
<td>20</td>
<td>100</td>
</tr>
<tr>
<td>Score:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Question 1 (30 points)
Let \( A \) be the matrix
\[
\begin{pmatrix}
1 & 2 & 0 & -2 \\
-1 & 0 & 2 & -2 \\
1 & 1 & -1 & 0 \\
-1 & 3 & 5 & -8
\end{pmatrix}
\]
and let \( T : \mathbb{R}^4 \to \mathbb{R}^4 \) be the linear transformation satisfying \( T(\vec{x}) = A\vec{x} \).

(a) (7 points) Use Gauss-Jordan elimination to compute the reduced row echelon form of \( A \).
Circle the leading 1s in \( \text{rref}(A) \).

Thus, \( \text{rref}(A) = \begin{pmatrix}
1 & 0 & -2 & 2 \\
0 & 1 & 1 & -2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \)
(leading 1s are circled.)

(b) (8 points) Find a basis for the kernel of \( T \). What is the dimension of the kernel of \( T \)?

Solve \( A\vec{x} = \vec{0} \).
\[
\begin{pmatrix}
1 & 2 & 0 & -2 \\
-1 & 0 & 2 & -2 \\
1 & 1 & -1 & 0 \\
-1 & 3 & 5 & -8
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]
Let \( x_4 = t, t \in \mathbb{R} \)
\[
x_3 = s, s \in \mathbb{R}
\]
\[
x_2 = 2t - s
\]
\[
x_1 = -2t + 2s
\]
Thus, \( \text{ker}(T) = \text{span} \left( \begin{pmatrix}
2 \\
1 \\
-1 \\
2
\end{pmatrix} \right) \).

A basis for \( \text{ker}(T) \) is \( \left( \begin{pmatrix}
-2 \\
1 \\
0 \\
2
\end{pmatrix}, \begin{pmatrix}
1 \\
0 \\
1 \\
0
\end{pmatrix} \right) \).
The dimension of \( \text{ker}(T) \) is 2.
(c) (8 points) Find a basis for the image of $T$. What is the dimension of the image of $T$?

From rank($A$), we see that columns 1 and 2 of $A$ are linearly independent and columns 3 and 4 are redundant.

Thus,

Basis of $\text{im}(A)$ is \[
\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.
\]

The dimension of $\text{im}(A)$ is 2.

(d) (7 points) Is $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ an invertible linear transformation? Justify your answer.

No, $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is not invertible. It has a non-trivial kernel.

(Other explanations are ok too.)
Question 2  (20 points)
Consider the linear system

\[
\begin{align*}
x + y - z &= 2 \\
x + 2y + z &= 3 \\
x + y + (k^2 - 5)z &= k
\end{align*}
\]

where \( k \) is an arbitrary constant. For which value(s) of \( k \) does this system have a unique solution? For which value(s) of \( k \) does the system have infinitely many solutions? For which value(s) of \( k \) is the system inconsistent?

Row reduce:

\[
\begin{bmatrix}
1 & 1 & -1 & 2 \\
1 & 2 & 1 & 3 \\
1 & 1 & k^2 - 5 & k
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & -1 & 2 \\
0 & 1 & 2 & 1 \\
0 & 0 & k^2 - 4 & k - 2
\end{bmatrix}
\]

- When \( k \neq 2, -2 \), we can divide by \( k^2 - 4 \) to uniquely solve for \( x, y, z \). Thus, there is a unique solution for any \( k \) \( \in \mathbb{R} \) except \( k = 2 \) and \( k = -2 \).
- When \( k = 2 \), we have:

\[
\begin{bmatrix}
1 & 1 & -1 & 2 \\
0 & 1 & 2 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

This yields infinitely many solutions.

- When \( k = -2 \), we have:

\[
\begin{bmatrix}
1 & 1 & -1 & 2 \\
0 & 1 & 2 & 1 \\
0 & 0 & 0 & -4
\end{bmatrix}
\]

There are no solutions to the linear system in this case.
Question 3  (30 points)
In each of the following parts, write down a $2 \times 2$ matrix $A$ that satisfies the given property.
If $A$ is invertible, write down the inverse. If $A$ is not invertible, explain why not.

(a) (7 points) $A\vec{x} = -3\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$.

$$A = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}$$

(b) (8 points) $A\vec{x}$ is the orthogonal projection of $\vec{x}$ onto the line $y = -2x$.

The line can be described as $\text{span}\left[ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right]$.

A unit vector in the direction of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is $\vec{u} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}$.

Then, $A(\vec{e}_1) = (\vec{e}_1 \cdot \vec{u})\vec{u} = \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ -\frac{2}{5} \end{bmatrix}$.

And, $A(\vec{e}_2) = (\vec{e}_2 \cdot \vec{u})\vec{u} = \frac{-2}{\sqrt{5}} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} \\ \frac{4}{5} \end{bmatrix}$.

So, $A = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix}$. Also, $\det(A) = 0$ so $A$ is not invertible.

(or argue geometrically)
(c) (8 points) \( A \) is the matrix of counterclockwise rotation by \( 60^\circ \).

\[
A = \begin{bmatrix}
\cos 60^\circ & -\sin 60^\circ \\
\sin 60^\circ & \cos 60^\circ
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & \frac{1}{2}
\end{bmatrix}
\]

\[
A^{-1} = \begin{bmatrix}
\cos(-60^\circ) & -\sin(-60^\circ) \\
\sin(-60^\circ) & \cos(-60^\circ)
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} & \frac{\sqrt{3}}{2} \\
-\frac{\sqrt{3}}{2} & \frac{1}{2}
\end{bmatrix}
\]

(d) (7 points) \( A \) is the matrix of the linear transformation described by “first rotate counterclockwise by \( 60^\circ \), and then project onto the line \( y = -2x \).”

\[
A = \begin{bmatrix}
\frac{1}{5} & -\frac{2}{5} \\
-\frac{2}{5} & \frac{4}{5}
\end{bmatrix} \begin{bmatrix}
\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & \frac{1}{2}
\end{bmatrix} = \begin{bmatrix}
\frac{1-2\sqrt{3}}{10} & \frac{-\sqrt{3}-2}{10} \\
\frac{-2+4\sqrt{3}}{10} & \frac{2\sqrt{3}+4}{10}
\end{bmatrix}
\]

Since the orthogonal projection is not invertible (it has a non-trivial kernel), the composition “first rotate by \( 60^\circ \), and then orthogonally project onto \( y = -2x \)” is not invertible.
Question 4  (20 points)
In each of the following questions, circle the correct answer. No justification is necessary.

(a) (5 points) If $A$ and $B$ are invertible $3 \times 3$ matrices, then $AB$ is invertible and $(AB)^{-1} = A^{-1}B^{-1}$.

True  False

(b) (5 points) Let $V$ be the set of all $(x, y, z) \in \mathbb{R}^3$ which satisfy the system of linear equations:

\[
2x - 3y + 7z = 0 \\
3x + y = 0 \\
x - 4y + z = 0
\]

Then, $V$ is a subspace of $\mathbb{R}^3$.

True  False

(c) (5 points) Let $A$ be an $5 \times 5$ matrix. If $\ker(A)$ is non-zero, then it is not possible for the columns of $A$ to form a basis.

True  False

(d) (5 points) If the linear system $Ax = \vec{b}$ is consistent, then the system $A^2 \vec{x} = \vec{b}$ must also be consistent.

True  False