Math 425 – Quiz 4

Question 1  (5 points)
20 people arrive separately to a dinner. Upon arrival, each person looks to see if he or she has any friends among those present. That person then either sits at the table of a friend or at an unoccupied table if none of those present is a friend. Assuming that each of the \( \binom{20}{2} \) pairs of people are, independently, friends with probability 0.5, find the expected number of occupied tables.

Solution:
Let \( X_i \) be a random variable that takes on value 0 if the \( i^{th} \) person to arrive sits at an existing table and 1 if he/she sits at a new table. Observe that
\[
E[X_i] = P(X_i = 1) = (1 - 0.5)^{i-1}.
\]
Then, the expected number of tables is
\[
E[X_1] + E[X_2] + \cdots + E[X_{20}] = 1 + 0.5 + 0.5^2 + \cdots + 0.5^{19} = \frac{1 - 0.5^{20}}{0.5}.
\]

Question 2  (5 points)
Let \( X \) be a random variable that measures the amount of time that a certain system functions. Suppose that the density function of \( X \) is given (in units of months) by
\[
f(x) = \begin{cases} 
Cxe^{-x/2} & : x > 0 \\
0 & : x \leq 0
\end{cases}
\]
(a) (2 points) Find \( C \) so that \( f(x) \) is a probability density function. (Hint: integration by parts)

Solution:
We need to find \( C \) such that
\[
1 = \int_0^\infty Cxe^{-x/2}dx = C \int_0^\infty xe^{-x/2}dx = 4C.
\]
Thus, \( C = 1/4 \).

(b) (2 points) What is the probability that the system functions for at least 6 months?

Solution:
The probability that the system functions for at least 6 months is
\[
\int_6^\infty \frac{1}{4}xe^{-x/2}dx = 4e^{-3}
\]

(c) (1 point) What is the probability that the system functions for exactly 4 months?

Solution:
The probability that the system functions for exactly 4 months is 0.