5.10 A circular current loop of radius $a$ carrying a current $I$ lies in the $x$-$y$ plane with its center at the origin.

a) Show that the only nonvanishing component of the vector potential is

$$A_\phi(\rho, z) = \frac{\mu_0 I a}{\pi} \int_0^\infty dk \, k \cos(kz) I_1(k\rho_<) K_1(k\rho_>),$$

where $\rho_<$ ($\rho>$) is the smaller (larger) of $a$ and $\rho$.

The vector potential may be obtained by

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'$$

where (for a circular current loop)

$$\vec{J}(\vec{x}') = I \delta(z') \delta(\rho' - a) \hat{\phi}'$$

in cylindrical coordinates. Note that to obtain the cylindrical components of $\vec{A}(\vec{x})$ we have to be careful to convert the basis vector $\hat{\phi}'$ at the point $x'$ to components at $x$. (This is because the basic vectors depend on position.) A bit of geometry gives

$$\hat{\phi}' = \hat{\rho} \sin(\phi - \phi') + \hat{\phi} \cos(\phi - \phi')$$

[Or, alternatively, we may choose the point $x$ to lie at $\phi = 0$, so that $\hat{\phi} = \hat{y}$ and $\hat{\rho} = \hat{x}$. Then it is straightforward to see that $\hat{\phi}' = \hat{y} \cos\phi' - \hat{x} \sin\phi' = \hat{\phi} \cos\phi' - \hat{\rho} \sin\phi'$. Using symmetry, we can see that only the $\hat{\phi}$ component of $\vec{A}$ is nonvanishing.]

The integral expression for the vector potential is then

$$\vec{A}(\vec{x}) = \frac{\mu_0 I a}{4\pi} \int \frac{\delta(z') \delta(\rho' - a) [\hat{\rho} \sin(\phi - \phi') + \hat{\phi} \cos(\phi - \phi')]}{|\vec{x} - \vec{x}'|} \rho' d\rho' d\phi' dz'$$

$$= \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} \frac{\hat{\rho} \sin(\phi - \phi') + \hat{\phi} \cos(\phi - \phi')}{|\vec{x} - \vec{x}'|} d\phi'$$

(1)

where the integrand in the second line is to be evaluated at $z' = 0$ and $\rho' = a$.

We now use the cylindrical Green’s function expressed as

$$\frac{1}{|\vec{x} - \vec{x}'|} = \frac{4}{\pi} \int_0^\infty dk \cos[k(z - z')] \left[ \frac{1}{2} I_0(k\rho_<) K_0(k\rho_>) \right.$$  

$$+ \sum_{m=1}^{\infty} \cos[m(\phi - \phi')] I_m(k\rho_<) K_m(k\rho_>) \right]$$
Note that the integral over $\phi'$ picks out the $m=1$ term in the sum. Furthermore, the $\hat{\rho}$ component drops out because $\sin(\phi - \phi')$ is orthogonal to $\cos(\phi - \phi')$, a result that could have been obtained by symmetry. We end up with

$$\vec{A}(\vec{x}) = \mu_0 Ia \frac{4}{4\pi} \hat{\phi} \int_0^\infty dk \cos(kz) I_1(k\rho_<) K_1(k\rho_>)$$

$$= \frac{\mu_0 Ia \hat{\phi}}{4\pi} \int_0^\infty dk \cos(kz) I_1(k\rho_<) K_1(k\rho_>)$$

b) Show that an alternative expression for $A_\phi$ is

$$A_\phi(\rho, z) = \mu_0 Ia \frac{2}{\rho} \int_0^\infty dk e^{-k|z|} J_1(ka) J_1(k\rho)$$

To obtain the alternative expression, we use the alternate form of the Greens’ function

$$\frac{1}{|\vec{x} - \vec{x}'|} = 2 \int_0^\infty dk e^{-k(z_>-z_<)} \left[ \frac{1}{2} J_0(k\rho) J_0(k\rho') + \sum_{m=1}^{\infty} \cos[m(\phi - \phi')] J_m(k\rho) J_m(k\rho') \right]$$

Since, for $z' = 0$, we have $z_> - z_< = |z|$, it is clear that when we stick this into (1) we end up with

$$\vec{A}(\vec{x}) = \frac{\mu_0 Ia \hat{\phi}}{2} \int_0^\infty dk e^{-k|z|} J_1(ka) J_1(k\rho)$$

c) Write down integral expressions for the components of magnetic induction, using the expressions of parts a) and b). Evaluate explicitly the components of $\vec{B}$ on the $z$ axis by performing the necessary integrations.

Since $\vec{B} = \vec{\nabla} \times \vec{A}$ and the only non-vanishing component of $\vec{A}$ is $A_\phi$, we end up with

$$B_\rho = -\partial_z A_\phi, \quad B_z = \frac{1}{\rho} \partial_\rho (\rho A_\phi)$$

The $z$ derivative is straightforward. For the $\rho$ derivative, on the other hand, we may use the Bessel equation identity

$$\frac{d}{dz} X_1(z) + \frac{1}{z} X_1(z) = X_0(z)$$

where $X_m$ denotes either $J_m$, $N_m$, $I_m$ or $K_m$. This gives, in particular

$$\frac{1}{\rho} \partial_\rho [\rho X_1(k\rho)] = kX_0(k\rho)$$
Hence, for the expression of \( a \) we find

\[
B_{\rho} = \frac{\mu_0 I a}{\pi} \int_0^\infty dk \, k \sin(kz) I_1(k\rho_<) K_1(k\rho_<)
\]

and

\[
B_z = \frac{\mu_0 I a}{\pi} \int_0^\infty dk \, k \cos(kz) \begin{cases} 
I_0(k\rho) K_1(ka) \\
I_1(ka) K_0(k\rho) 
\end{cases}
\]

where the top line holds for \( \rho < a \), while the bottom line holds for \( \rho > a \).

Similarly, the vector potential of \( b \) yields the magnetic induction

\[
B_{\rho} = -\frac{\mu_0 I a}{2} \operatorname{sgn}(z) \int_0^\infty dk \, k e^{-k|z|} J_1(k\rho) J_1(ka)
\]

and

\[
B_z = \frac{\mu_0 I a}{2} \int_0^\infty dk \, k e^{-k|z|} J_0(k\rho) J_1(ka)
\]

The \( z \) axis corresponds to \( \rho = 0 \). In this case, it is easy to see that \( B_{\rho} = 0 \) (a result demanded by symmetry) follows from the result that either \( J_1(0) = 0 \) or \( I_1(0) = 0 \). For the \( B_z \) component, we take the representation of part \( b \). Noting that \( J_0(0) = 1 \), we end up with

\[
B_z(\rho = 0) = \frac{\mu_0 I a}{2} \int_0^\infty dk \, k e^{-k|z|} J_1(ka)
\]

\[
= \frac{\mu_0 I a}{2} \frac{a}{(z^2 + a^2)^{3/2}}
\]

\[
= \frac{\mu_0 I a^2}{2(z^2 + a^2)^{3/2}}
\]

which agrees with the elementary result for a current loop on axis. [This integral was performed by noting that it is a Laplace transform \( \mathcal{L}\{t J_1(at)\} \), which in turn is the derivative \( -d/ds \) of the transform \( \mathcal{L}\{J_1(at)\} \). The Laplace transform of a Bessel function can be looked up, with the result \( \mathcal{L}\{J_n(at)\} = a^{-n}(\sqrt{s^2 + a^2} - s)^n/\sqrt{s^2 + a^2} \).

5.14 A long, hollow, right circular cylinder of inner (outer) radius \( a \) (\( b \)), and of relative permeability \( \mu_r \), is placed in a region of initially uniform magnetic-flux density \( \vec{B}_0 \) at right angles to the field. Find the flux density at all points in space, and sketch the logarithm of the ratio of the magnitudes of \( \vec{B} \) on the cylinder axis to \( \vec{B}_0 \) as a function of \( \log_{10} \mu_r \) for \( a^2/b^2 = 0.5, 0.1 \). Neglect end effects.

For a long cylinder (neglecting end effects) we may think of this as a two-dimensional problem. Since there are no current sources, we use a magnetic scalar
potential $\Phi_M$ which must be harmonic in two dimensions. Since $\vec{H} = -\vec{\nabla}\Phi_M$, we orient the uniform magnetic field $H_0$ along the $+x$ axis and write

$$\Phi_M(\rho, \phi) = \begin{cases} (-H_0 \rho + \sum \frac{\alpha}{\rho}) \cos \phi, & \rho > b \\ (\beta \rho + \frac{\gamma}{\rho}) \cos \phi, & a < \rho < b \\ \delta \rho \cos \phi, & \rho < a \end{cases}$$

(2)

Of course, the general harmonic expansion would be of the form $(A_m \rho^m + B_m \rho^{-m}) \cos m\phi + (C_m \rho^m + D_m \rho^{-m}) \sin m\phi$. However here we have already used the shortcut that all matching conditions for $m \neq 1$ lead to homogeneous equations admitting only a trivial (zero) solution.

The magnetostatic boundary conditions demand that $H_\phi$ and $B_\rho$ are continuous at both $\rho = a$ and $\rho = b$. The magnetic field (and magnetic induction) components are

$$H_\phi = -\frac{1}{\rho} \partial_\phi \Phi_M = \begin{cases} (-H_0 + \frac{\alpha}{\rho^2}) \sin \phi, & \rho > b \\ (\beta + \frac{\gamma}{\rho^2}) \sin \phi, & a < \rho < b \\ \delta \sin \phi, & \rho < a \end{cases}$$

and

$$B_\rho = \mu \partial_\rho \Phi_M = \begin{cases} \mu_0 (-H_0 - \frac{\alpha}{\rho^2}) \cos \phi, & \rho > b \\ \mu (\beta - \frac{\gamma}{\rho^2}) \cos \phi, & a < \rho < b \\ \mu_0 \delta \cos \phi, & \rho < a \end{cases}$$

The resulting matching conditions at $a$ and $b$ are

$$-H_0 + \frac{\alpha}{b^2} = \beta + \frac{\gamma}{b^2}, \quad -H_0 - \frac{\alpha}{b^2} = \mu_r \left(\beta - \frac{\gamma}{b^2}\right)$$

$$\beta + \frac{\gamma}{a^2} = \delta, \quad \beta - \frac{\gamma}{a^2} = \frac{1}{\mu_r}$$

where $\mu_r = \mu/\mu_0$. These equations may be solved to yield

$$\alpha = \Delta^{-1}(\mu_r - \mu_r^{-1})(b^2 - a^2)H_0$$

$$\beta = -2\Delta^{-1}(1 + \mu_r^{-1})H_0$$

$$\gamma = -2\Delta^{-1}(1 - \mu_r^{-1})a^2H_0$$

$$\delta = -4\Delta^{-1}H_0$$

where

$$\Delta = (1+\mu_r)(1+\mu_r^{-1})+(1-\mu_r)(1-\mu_r^{-1}) \left(\frac{a}{b}\right)^2 = \frac{1}{\mu_r} \left[ (\mu_r + 1)^2 - (\mu_r - 1)^2 \left(\frac{a}{b}\right)^2 \right]$$

The magnetic scalar potential is then given by (2) with the above values of the coefficients. We see that the magnetic induction for $\rho < a$ is uniform, pointed
along the same direction as $\vec{B}_0$. The other two regions contain a dipole field in addition a uniform component.

Since $\vec{H} = -\vec{\nabla}\Phi_M = -\delta \hat{x}$ for $\rho < a$, the ratio of $\vec{B}$ on axis ($\rho = 0$) to $\vec{B}_0$ is given by

$$\frac{B}{B_0} = 4\Delta^{-1} = \frac{4}{(1 + \mu_r)(1 + \mu_r^{-1}) + (1 - \mu_r)(1 - \mu_r^{-1})(a/b)^2}$$

This may be plotted as follows

\[ B / B_0 \]

\[ (a / b)^2 = 0.5 \]

\[ (a / b)^2 = 0.1 \]

\[ \log_{10} \mu_r (a/b)^2 = 0.1 \]

5.17 A current distribution $\vec{J}(\vec{x})$ exists in a medium of unit relative permeability adjacent to a semi-infinite slab of material having relative permeability $\mu_r$ and filling the half-space, $z < 0$.

a) Show that for $z > 0$ the magnetic induction can be calculated by replacing the medium of permeability $\mu_r$ by an image current distribution, $\vec{J}^*$, with components,

\[
\left(\frac{\mu_r - 1}{\mu_r + 1}\right) J_x(x, y, -z), \quad \left(\frac{\mu_r - 1}{\mu_r + 1}\right) J_y(x, y, -z), \quad -\left(\frac{\mu_r - 1}{\mu_r + 1}\right) J_z(x, y, -z)
\]

We will end up solving parts a) and b) simultaneously. We start, however, by defining the reflection (Parity) operator $P : z \rightarrow -z$ so that

$$P : (x, y, z) \rightarrow (x, y, -z)$$

On the right ($z > 0$), we assume the magnetic induction is generated by both the original current $\vec{J}$ (contained entirely on the right) and an image current $\vec{J}^*$ (contained entirely on the left). Thus

$$\vec{B}_R(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\left(\vec{J}(\vec{x}') + \vec{J}^*(\vec{x}')\right) \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} \, d^3x'$$

By changing variables $z' \rightarrow -z'$ in the $\vec{J}^*$ term, we may restrict this volume integral to $z' > 0$

$$\vec{B}_R(\vec{x}) = \frac{\mu_0}{4\pi} \int_{z' > 0} \left(\frac{\vec{J}(\vec{x}') \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} + \frac{\vec{J}^*(P\vec{x}') \times (\vec{x} - P\vec{x}')}{|\vec{x} - P\vec{x}'|^3}\right) \, d^3x' \quad (3)$$
On the left \((z < 0)\), we assume the magnetic induction is generated by a current of the same form as the original \(\vec{J}\), but with possibly modified strength (because of the change of permeability). Given a modified current \(\lambda \vec{J}\) and permeability \(\mu\), we write

\[
\vec{B}_L(x) = \mu \lambda \int_{z' > 0} \frac{\vec{J}(x') \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} \, d^3 x'
\]  

(4)

Our aim is now to match the left and right magnetic field and magnetic induction. More precisely, at \(z = 0\), both \(H_x\) and \(H_y\) (the parallel components) must be continuous, and \(B_z\) (the perpendicular component) must also be continuous. To perform this matching, we first note that the norms \(|\vec{x} - \vec{x}'|\) and \(|\vec{x} - P\vec{x}'|\) are identical at \(z = 0\). (They are both equal to \(\sqrt{(x - x')^2 + (y - y')^2 + z'^2}\).) Thus all denominators are the same, and we deduce that the numerators of (3) and (4) must be matched as appropriate. For \(B_z\), we have

\[
(J_x + J_x^*)(y' - y) - (J_y + J_y^*)(x - x') = \mu_r \lambda (J_x(y' - y) - J_y(x - x'))
\]

where any component of \(\vec{J}^*\) is understood to have argument \(P\vec{x}\). For \(H_x\) and \(H_y\) matching, we find

\[
-(J_y - J_y^*)z' - (J_z + J_z^*)(x - x') = \lambda (-J_yz' - J_z(x - x'))
\]

\[
(J_z + J_z^*)(x - x') + (J_x - J_x^*)z' = \lambda (J_z(x - x') + J_xz')
\]

Since these equations hold for all values of \((x, y)\), they separate into

\[
\lambda J_y = J_y - J_y^* \quad \lambda J_z = J_z + J_z^*
\]

\[
\lambda J_z = J_z + J_z^* \quad \lambda J_x = J_x - J_x^*
\]

\[
\mu_r \lambda J_x = J_x + J_x^* \quad \mu_r \lambda J_y = J_y + J_y^*
\]

These equations may be solved to yield

\[
J_x^* = (1 - \lambda)J_x, \quad J_y^* = (1 - \lambda)J_y, \quad J_z = -(1 - \lambda)J_z
\]

provided \(\mu_r \lambda - 1 = 1 - \lambda\), or \(\lambda = 2/(\mu_r + 1)\). This may be given in a more concise form using the reflection operator

\[
\vec{J}^*(\vec{x}) = (1 - \lambda)P\vec{J}(P\vec{x}) = \frac{\mu_r - 1}{\mu_r + 1}P\vec{J}(P\vec{x})
\]

b) Show that for \(z < 0\) the magnetic induction appears to be due to a current distribution \([2\mu_r/(\mu_r + 1)]J\) in a medium of unit relative permeability.

From the expression (4) for \(\vec{B}_L\), the magnetic induction appears to be due to a current \(\lambda \vec{J} = [2/(\mu_r + 1)]\vec{J}\) in a medium of permeability \(\mu\). This is equivalent
5.19 A magnetically “hard” material is in the shape of a right circular cylinder of length $L$ and radius $a$. The cylinder has a permanent magnetization $M_0$, uniform throughout its volume and parallel to its axis.

a) Determine the magnetic field $\vec{H}$ and magnetic induction $\vec{B}$ at all points on the axis of the cylinder, both inside and outside.

We use a magnetic scalar potential and the expression

$$\Phi_M = -\frac{1}{4\pi} \int_V \frac{\nabla \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3 x' + \frac{1}{4\pi} \oint_S \frac{\hat{n}' \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} d\alpha'$$

Orienting the cylinder along the $z$ axis, we take a uniform magnetization $\vec{M} = M_0 \hat{z}$. In this case the volume integral drops out, and the surface integral only picks up contributions on the endcaps. Thus

$$\Phi_M = \frac{M_0}{4\pi} \left[ \int_{\text{top}} \frac{1}{|\vec{x} - \vec{x}'|} d\alpha' - \int_{\text{bottom}} \frac{1}{|\vec{x} - \vec{x}'|} d\alpha' \right]$$

where ‘top’ and ‘bottom’ denote $z = \pm L/2$, and the integrals are restricted to $\rho < a$. On axis ($\rho = 0$) we have simply

$$\Phi_M(z) = \frac{M_0}{4\pi} \int \left( \frac{1}{\sqrt{\rho^2 + (z-L/2)^2}} - \frac{1}{\sqrt{\rho^2 + (z+L/2)^2}} \right) \rho d\rho d\phi$$

$$= \frac{M_0}{4} \int_0^{a^2} \left( \frac{1}{\sqrt{\rho^2 + (z-L/2)^2}} - \frac{1}{\sqrt{\rho^2 + (z+L/2)^2}} \right) d\rho^2$$

$$= \frac{M_0}{2} \left[ \sqrt{a^2 + (z-L/2)^2} - \sqrt{a^2 + (z+L/2)^2} - |z - L/2| + |z + L/2| \right]$$

On axis, the field can only point in the $z$ direction. It is given by

$$H_z = -\partial_z \Phi_M = -\frac{M_0}{2} \left[ \frac{z - L/2}{\sqrt{a^2 + (z-L/2)^2}} - \frac{z + L/2}{\sqrt{a^2 + (z+L/2)^2}} - \text{sgn}(z-L/2) + \text{sgn}(z+L/2) \right]$$

Note that the last two terms cancel when $|z| > L/2$, but add up to 2 inside the magnet. Thus we may write

$$H_z = -\frac{M_0}{2} \left[ \frac{z - L/2}{\sqrt{a^2 + (z-L/2)^2}} - \frac{z + L/2}{\sqrt{a^2 + (z+L/2)^2}} + 2 \Theta(L/2 - |z|) \right]$$
where \( \Theta(\xi) \) denotes the unit step function, \( \Theta = 1 \) for \( \xi > 0 \) (and 0 otherwise). The magnetic induction is obtained by rewriting the relation \( \vec{H} = \vec{B}/\mu_0 - \vec{M} \) as \( \vec{B} = \mu_0(\vec{H} + \vec{M}) \). Since the magnetization is only nonzero inside the magnet [ie \( M_z = M_0 \Theta(L/2 - |z|) \)], the addition \( \vec{H} + \vec{M} \) simply removes the step function term. We find

\[
B_z = \mu_0(H_z + M_z) = -\frac{\mu_0 M_0}{2} \left[ \frac{z - L/2}{\sqrt{a^2 + (z - L/2)^2}} - \frac{z + L/2}{\sqrt{a^2 + (z + L/2)^2}} \right]
\]

b) Plot the ratios \( \vec{B}/\mu_0 M_0 \) and \( \vec{H}/M_0 \) on the axis as functions of \( z \) for \( L/a = 5 \).

The \( z \) component of the magnetic field looks like

\[ \frac{H}{M_0} \]

while the \( z \) component of the magnetic induction looks like

\[ \frac{B}{\mu_0 M_0} \]

Note that \( B_z \) is continuous, while \( H_z \) jumps at the ends of the magnet. This jump may be thought of as arising from effective magnetic surface charge.