Problem Set 3  
Due Friday, January 30

1. Consider a spin 1/2 atom in a state $|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$ in a system quantized along the $z$-axis. Suppose there is a magnetic field $\mathbf{B}$ having polar angles $(\theta, \phi)$. Rotate the system so the magnetic field is along the $z$-axis and calculate the components of $|\psi\rangle$ after this active rotation [Hint: show that the active rotation is given by Euler angles $(\alpha, \beta, \gamma) = (\pi/2, -\theta, -\phi)$]. If $a = 1$ and $b = 0$ show that the result agrees with that given on page 180 of the coursepack notes with $\phi = \pi/2, \theta \rightarrow \phi$.

2-3. Consider a spin one particle having standard components

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \quad S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$  

Show that for an infinitesimal rotation around the $z$-axis, that the components of $\mathbf{S}$ transform as a vector (this is true for any $3 \times 3$ matrices obeying the standard commutation relations). On the other hand, show that the components of the spin state

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_0 \\ \psi_{-1} \end{pmatrix}$$

do not transform as a cartesian vector. Thus, in the standard $|s m\rangle$ basis, the components of $\psi$ do not transform as a cartesian vector. However, there is a basis in which the components of $\psi$ do transform as a cartesian vector. To see this explicitly show that under the unitary transformation

$$\psi' = \begin{pmatrix} \psi_x \\ \psi_y \\ \psi_z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_0 \\ \psi_{-1} \end{pmatrix}$$

the spin $\mathbf{S}$ transforms as

$$S'_x = i\hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}; \quad S'_y = i\hbar \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}; \quad S'_z = i\hbar \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and that the components of $\psi'$ transform as a vector under the infinitesimal rotation $U = \exp[-i S'/\hbar]$. Actually, both $\psi'$ and $\psi$ transform as vectors under rotation. It is just that different definitions of the transformation properties are used for standard and cartesian
components of a vector.

4. Consider the 2-D isotropic harmonic oscillator with Hamiltonian 
\[ H = \frac{\hbar \omega}{2} (\eta^2 + \xi^2), \]
where \( \eta \) and \( \xi \) are dimensionless variable satisfying \([\xi_x, \eta_x] = i; [\xi_y, \eta_y] = i\). Prove that the Hamiltonian can be written in the form 
\[ H = \hbar \omega \sqrt{(4K^2 + 1)}, \]
where
\[
K_1 = \frac{\eta_x^2 - \eta_y^2}{4} + \frac{\xi_x^2 - \xi_y^2}{4}; \\
K_2 = \frac{\eta_x \eta_y}{2} + \frac{\xi_x \xi_y}{2}; \\
K_3 = \frac{\xi_x \eta_y - \xi_y \eta_x}{2}. 
\]
Show that the \( K \)'s satisfy the standard commutation relations for angular momentum matrices. Use this fact to obtain the allowed energy levels for the oscillator and to prove that the underlying symmetry group is \( SU(2) \). How does this explain the degeneracy of the levels?

5. Use the program on page 299 of the coursepack to calculate the rotation matrices for \( J=1/2, J=1, \) and \( J=3/2 \).