Problem Set 1
Due Tuesday, January 14

1. Problem 1 from "Problems" starting on page 349 of coursepack.
2-4. Consider the differential equation $\dot{y} = Ay$, where $y$ is a column vector and $A$ is a constant matrix.

- Show, by direct substitution that a solution to this equation is $y(t) = e^{At}y(0)$.

In the following parts, take

$$A = -i \begin{pmatrix} -a & b \\ b & a \end{pmatrix}; \quad y(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}; \quad y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

- Solve the differential equation directly by assuming a solution of the form $y_i(t) = B_i e^{\lambda_i t}$, $i = 1, 2$.

- Solve the equations using the identity

$$e^{-i\hat{n} \cdot \sigma} = 1 \cos \theta - i \hat{n} \cdot \sigma \sin \theta$$

where $\hat{n}$ is a unit vector and $\sigma$ is a vector having matrix components

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- Solve the equation using the MatrixExp[{{I a,-I b},{-I b,-I a}}] function of Mathematica.

- Find a matrix $T$ such that $TAT^\dagger = \Lambda$, where $\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}$ is a diagonal matrix. Prove that

$$y(t) = T^\dagger e^{\Lambda t} Ty(0) = T^\dagger \begin{pmatrix} e^{\Lambda_1 t} & 0 \\ 0 & e^{\Lambda_2 t} \end{pmatrix} Ty(0),$$

and evaluate this explicitly.

Show that all your results give the same solution. Note that the last method can be used for matrices of any dimension.