Collisions

Colloquially a collision occurs when two objects hit one another. This has many of the characteristics of what a collision means in the study of physics, but we can lend some precision to the definition. A collision occurs when two, or more, objects interact with one another over a relatively brief period of time. Is a planet orbiting a star a collision? No, with a bound orbit the time of interaction is indefinitely long. What about a comet that enters the solar system, rounds the Sun and exits to never return. Is that a collision? Yes. There are two particles interacting, gravitationally, for a relatively short length of time. For this situation you might have two questions, the first is that it often takes several months for a comet to pass by the Sun, why is this a short time? The answer is short compared to what? That comet has been traveling in a straight line with constant momentum for more than a thousand years. Then it comes near our solar system and for several months it experiences an attractive force for that significantly changes its momentum. After rounding the Sun it again travels with constant momentum for many many years. So the time over which the momentum changes is short compared to the length of the time of the trajectory. The second question that you might have is how can an attractive force cause a collision? The sign of the force is immaterial, either an attractive or repulsive force will cause a change in momentum and that is all that is necessary.

Writing Newton’s law \( \frac{dp}{dt} = F \) we can integrate this from a time before the collision to a time after. The specific times do not matter as long a they are when the force is near zero. Rewriting the equation as \( dp = F dt \) and integrating gives \( \int_{Before}^{After} dp = \int_{Before}^{After} F dt \). The left hand side is just the change in momentum \( \Delta p = p_{After} - p_{Before} \). There is nothing that we can do to simplify the right hand side, but we can give this quantity a name. It is called the impulse and often given the symbol \( J = \int_{Before}^{After} F dt \). It is important to remember that \( J \) is a vector quantity.

It is useful to look at an example. Consider a collision in one dimension so that the force can be sketched on a single axis, figure on right.

The force on one object is very small both before and after the collision. In between it increases. The impulse \( J \) is equal to the area under the curve, shown in red. If the start and end of the collision occurs at the marked times then the average force is defines the rectangle with the same area, \( Area = F_{ave} \Delta t \), as the force curve. For this particular
collision you might want to move the Before and After times closer so that $\Delta t$ is smaller. This would give a larger average force.

For a two-dimensional collision there could be forces in both the x and y directions and so there would be two sketches like this. Consider the isolated collision between the proton and He$_4$ nucleus. You have previously seen the collision, shown from the laboratory frame. The proton travels toward the He$_4$ nucleus from left to right, positive x velocity, and is scattered towards the positive y direction.

Below are graphs of the x and y components of the force that are placed on the proton during the collision from the Coulomb interaction. Looking at the x-component of the force, the start of the interaction has the force in the negative x direction, as the proton passes by the nucleus the force in the x direction goes through zero and becomes positive. So in the x-direction the proton slows down and then speeds up. Eye-ball ing the curve it looks like the net area under the curve is negative and that the x-momentum of the proton is reduced during the collision. Of course, we could record the change in the momentum numerically and get an accurate answer.

Turning to the y-direction, this component of the force is always positive meaning that the proton will gain momentum in the positive y-direction. This is clearly evident in the bending of its trajectory upward. One thing that is a surprise is the magnitude of the peak interaction force, 150 N! This is 33 lbs on an object with a mass of $10^{-27}$ kg. Of course, the interaction time is very short, less that $10^{-21}$ sec. But, this is still a surprisingly large force.

How would these graphs differ if we had chosen to look at the He$_4$ nucleus? Remember that the mass of the nucleus is four times larger. Does the mass matter? NO! What matters is Newton’s third law. The He$_4$ nucleus experiences an equal and opposite force as compared to the proton. This means that the graphs would simply be reflected with $\vec{F} \Rightarrow -\vec{F}$. Being an isolated system the impulse on the He$_4$
nucleus is the negative of the impulse on the proton, \( \vec{J}_{He} = -\vec{J}_{proton} \). This assures that the net change in the momentum of the isolated system is zero.

We have not yet explicitly discussed kinetic energy, but you have no doubt learned that the kinetic energy of a particle is \( KE = \frac{1}{2}mv^2 = \frac{p^2}{2m} \). It is interesting to think about what happens to the total kinetic energy of a system when a collision occurs. If there are ways for some of the energy to be lost from the system then the final kinetic energy of the system can be reduced. For macroscopic objects the lost mechanical energy often is converted into heat. This is the typical situation for many collisions between macroscopic objects. When two cars collide, a large amount of kinetic energy is used to deform the materials and is eventually turned into heat. When kinetic energy is lost, the collision is called inelastic.

Under special conditions the kinetic energy does not change during a collision. This is called an elastic collision. The collision between two billiard balls is close to elastic. Often the collisions between microscopic particles are elastic. You could check to see if the p-He collision is elastic. It is. Never assume a collision is elastic, this is a special case.

Now for the very special case of one-dimensional elastic collisions in an isolated system we can find an explicit solution.

You must conserve both momentum and kinetic energy for an elastic collision. Therefore

\[
m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f} \quad \text{and} \quad \frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2.\]

Note that since this is a one dimensional collision the momentum conservation expression just uses scalars with positive and negative sign to indicate direction.

With the masses and initial velocities as given quantities you can solve for the unknown final velocities.

\[
v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right)v_{2i} \quad \text{and} \quad v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)v_{2i}.
\]

These solutions have several interesting special cases. If the two particles have equal mass, then \( v_{1f} = v_{2i} \) and
\( v_{2f} = v_{ii} \). This shows that the particles simply exchange velocities. A second interesting case is when \( m_2 \) is at rest. Then the expressions simplify to

\[
v_{1f} = \left( \frac{m_1 - m_2}{m_1 + m_2} \right) v_{ii} \quad \text{and} \quad v_{2f} = \left( \frac{2m_1}{m_1 + m_2} \right) v_{ii}
\]

If now again the two masses are equal then \( v_{1f} = 0 \) and \( v_{2f} = v_{ii} \). An example of this is shown on a pool table when the cue ball squarely bits another ball. The cue ball stops and the struck ball moves off in a straight line with the original velocity of the cue ball. A second interesting limit is if \( m_2 \) is much greater than \( m_1 \). This is the case of a Ping-Pong ball hitting a bowling ball. The velocity of the Ping-Pong ball is reversed and the velocity of the bowling ball remains zero.

A completely different situation involves a totally inelastic collision. In this case the two masses stick together after they have collided. Momentum is still conserved, but kinetic energy is lost in the collision.

\[
\begin{align*}
\text{Initial state before collision} & \quad \text{final state after collision} \\
\begin{array}{c}
m_1 \\
v_{ii}
\end{array} & \quad \begin{array}{c}
m_2 \\
v_{2i}
\end{array} & \quad \begin{array}{c}
M = m_1 + m_2 \\
V_f
\end{array}
\end{align*}
\]

Momentum conservation gives \( m_1 v_{ii} + m_2 v_{2i} = MV_f \). Therefore \( V_f = \frac{m_1 v_{ii} + m_2 v_{2i}}{M} \). If you are in the center of momentum frame then the total momentum is zero and the two particle collide and are at rest.