Conservation of momentum

So far we have been dealing with one particle systems that are experiencing external forces. Examples include the mass on a spring oscillating, skydiver falling or moon in orbit. In each of these cases there was a force external to the system that was responsible for the change in momentum of the object.

In this class we are going to shift from one particle systems experiencing an external force to a multiparticle system that is isolated from external forces. A good example of this is a binary star system. Such star systems are actually much more common than isolated stars such as our Sun. In any case, there are two objects interacting through a force, gravity. What can be said in general about the system?

Let’s start with a free-body diagram:

![Free-body diagram](image)

There are two forces in the system: the force that is placed on $m$ by $M$, $\vec{F}_{mM}$ and the force that is placed on $M$ by $m$, $\vec{F}_{Mm}$. These are equal and opposite forces, as can seen from the gravitational force law, or more generally from Newton’s third law. Given these forces the momentum of each mass must be changing. $\frac{d\vec{p}_m}{dt} = \vec{F}_{mM}$ and $\frac{d\vec{p}_M}{dt} = \vec{F}_{Mm}$. If the forces are known then it is easy to find the motion of the bodies. However, sometimes the interaction force is not known, or so complicated, that the equations of motion cannot be determined directly. In these situations it is extremely useful to consider the total momentum of the system, $\vec{P}_{total} = \vec{p}_m + \vec{p}_M$. Looking at the time rate of change of the total momentum gives:

$$\frac{d\vec{P}_{total}}{dt} = \frac{d\vec{p}_m}{dt} + \frac{d\vec{p}_M}{dt} = \vec{F}_{mM} + \vec{F}_{Mm} = 0.$$  This is a very important result: the total momentum is constant. This is true because the system is isolated and all the forces come as Newton’s third law reaction pairs. When you sum up the changes in the individual momenta they cancel. This derivation can trivially be extended to any number of particles.

The law of conservation of linear momentum states that for an isolated system the total momentum is a constant in time. Stated mathematically the total momentum defined as
\[ \vec{p}_{\text{total}} = \sum_i \vec{p}_i = \sum_i \gamma m \vec{v}_i \] satisfies \[ \frac{d\vec{p}_{\text{total}}}{dt} = 0. \] This is true for any isolated system, with any number of particles, containing any forces. It is this generality that makes the law of Conservation of Linear Momentum so powerful.

The thrust of a rocket is an excellent example of how to use the law of conservation of momentum for an isolated system. Consider an isolated (deep space) rocket of mass \( M \), at time \( t \), moving with velocity \( \vec{v} \), as shown in the figure on the left. An instant of time \( \Delta t \) later the situation looks like the figure on the right.

Since the system is isolated we know that \( \frac{d\vec{p}_{\text{total}}}{dt} = 0 \) this means that for a short time interval \( \Delta t \) that we can write \( 0 = \frac{\Delta \vec{p}}{\Delta t} = \frac{\vec{p}_{t+\Delta t} - \vec{p}_t}{\Delta t} \) where \( \vec{p}_{t+\Delta t} \) is the momentum for the system on the right and \( \vec{p}_t \) is the momentum for the system on the left. Examining the figures shows that \( \vec{p}_{t+\Delta t} = (M - \Delta M)(\vec{v} + \Delta \vec{v}) + \Delta M \vec{u} \) and \( \vec{p}_t = M \vec{v} \). Using conservation of momentum gives

\[
0 = \frac{\Delta \vec{p}}{\Delta t} = \frac{[(M - \Delta M)(\vec{v} + \Delta \vec{v}) + \Delta M \vec{u}] - [M \vec{v}]}{\Delta t} = \frac{M \vec{v} + M \Delta \vec{v} - \Delta M \vec{v} - \Delta M \Delta \vec{v} + \Delta M \vec{u} - M \vec{v}}{\Delta t}.
\]

Taking the \( \lim_{\Delta t \to \infty} \), dropping the term \( \Delta M \Delta \vec{v} \) that is proportional to \( (\Delta t)^2 \), and rearranging leads to

\[
M \frac{d\vec{v}}{dt} = (\vec{u} - \vec{v}) \frac{dM}{dt} \quad \text{where} \quad \frac{\Delta M}{\Delta t} = -\frac{dM}{dt}
\]

because the change in mass with respect to time is negative. The right-hand term only depends on the rocket’s present properties, the left-hand allows us to predict where the rocket will be in the future. Let’s examine the right-hand side more closely. \( \frac{dM}{dt} \) is the negative of the mass of the gas per second being exhausted out the back of the rocket. This is controlled by the design of the rocket. What about the term \( (\vec{u} - \vec{v}) \)? Looking at the figures above, this is just the velocity of the exhaust gas relative to the rocket. If riding on the rocket you see gas moving backwards.
away from you with this velocity; it is a vector that points to the left in the figure above. You should draw it in.

We will define \( \vec{v}_{rel} = \vec{u} - \vec{v} \), this is the exhaust velocity of the gas and is also a designed property of the rocket, via nozzle size and combustion rate etc. With this definition the equation becomes,

\[
M \frac{d\vec{v}}{dt} = \vec{v}_{rel} \frac{dM}{dt}
\]

The two terms on the right are properties of the rocket and the product has the dimensions of a force. This product is called the rocket “thrust”. The left hand side is the mass times the acceleration of the rocket. This equation is often called the first rocket equation.

Let’s examine your homework problem, in it the thrust is 100 N and assuming a linear burn rate, or loss of mass,

\[
\left| \frac{dM}{dt} \right| = 0.3 \text{kg/5 sec} = 0.06 \text{kg/s}.
\]

From this the exhaust velocity of the gas can be determined from

\[
|\vec{v}_{rel}| = \frac{F_{thrust}}{\left| \frac{dM}{dt} \right|} = 1667 \text{ m/s}.
\]

Typically, a rocket engineer designs an engine to have an initial mass, burn rate, exhaust velocity and burn time. With these parameters she can impart a force over a time (an impulse) to give the rocket the desired change in momentum.

A little more can be done to analyze the rocket equation, \( M \frac{d\vec{v}}{dt} = \vec{v}_{rel} \frac{dM}{dt} \). Multiply each side by \( dt \), rearranging the mass terms so they are all on the right-hand side yields:

\[
\int_{t_{\text{initial}}}^{t_{\text{final}}} d\vec{v} = \vec{v}_{rel} \int_{t_{\text{initial}}}^{t_{\text{final}}} \frac{dM}{M}
\]

and then integrate to produce the second rocket equation.

\[
\vec{v}(t_{\text{final}}) - \vec{v}(t_{\text{initial}}) = \vec{v}_{rel} \ln \frac{M_f}{M_i} = -\vec{v}_{rel} \ln \frac{M_i}{M_f}
\]

This allows you to find the velocity change of a rocket only knowing the exhaust velocity and the initial and final masses. The time of the burn does not enter into the equation. Therefore small engines that burn a long time are equivalent to large engines that only burn briefly. This result is true for rockets not experiencing gravitational forces. For a rocket taking off from Earth this result is not correct. In that situation you want to get rid of as much mass as possible, as quickly as possible. That is why rockets are staged with very large motors on the first stage. You will explore this a bit in part 2d of your homework.