Keplerian Orbits

If two otherwise isolated particles interact through a $\frac{1}{r^2}$ force law their trajectories can be reduced to conic sections. This is called Kepler’s problem after Johannes Kepler who studied planetary motion. If one of the particles is so massive that it can be approximated as fixed, then the other particle’s motion will be a conic section. The trajectories can be hyperbolic, parabolic, and elliptical. Linear and circular cases are special examples of these orbits. These trajectories can be classified as either “open” or “closed” by whether the particle visits infinity and therefore never returns (because the force of interaction has vanished). On an open orbit, parabolic and hyperbolic, the particles reach the point of closest approach only once and then separate forever after, i.e. one traversal; whereas for a closed orbit, elliptical, the motion is periodic repeating over and over identically.

It is interesting to note that there are only two mathematical types of forces that produce such motion that exactly repeat over-and-over. They are the linear Hooke’s law and the inverse square force law. As such, these are often considered the two most important problems in all of classical mechanics. All other forms of interaction force produce trajectories that either don’t trace out the same path (though don’t separate infinitely) or just move apart forever.

Let’s take a look at a comet entering our solar system (assume mass comet $<<$ mass sun). At some position $\vec{r}$ it has a momentum $\vec{p}$. The force that is exerted on the comet by the sun is $\vec{F}_{C,S} = -G \frac{Mm}{r_{C,S}^2} \vec{r}_{C,S}$. It is very useful to decompose this force into components parallel ($\hat{p}$) and perpendicular ($\hat{n}$) to the comet’s momentum (shown in red):
The $F_{\text{parallel}}$ ($\vec{F}_p$) causes a change in the speed of the Comet, while $F_{\text{perpendicular}}$ ($\vec{F}_\perp$) causes a change in direction of the momentum. The osculatory or “kissing circle” gives the radius of curvature at the location C. It is the circle that “best” approximates the curvature of the trajectory at point C.

The comet momentum is defined as $\hat{\rho} = |\hat{\rho}| \hat{\rho}$, taking the derivative with respect to time using the product rule gives,

$$\frac{d\hat{\rho}}{dt} = \frac{d|\hat{\rho}|}{dt} \hat{\rho} + |\hat{\rho}| \frac{d\hat{\rho}}{dt}$$

Therefore equating the parallel and perpendicular components of Newton’s equation yields,

$$\vec{F}_p = \frac{d|\hat{\rho}|}{dt} \hat{\rho} \quad \text{(linear motion) and} \quad \vec{F}_\perp = |\hat{\rho}| \frac{d\hat{\rho}}{dt} \quad \text{(curvature)}$$

So just what is $\frac{d\hat{\rho}}{dt}$, the change in time of the unit vector of momentum?

Here is a diagram:
Consider the comet at two times, initial $i$ and final $f$ where $\Delta t = t_f - t_i$. The comet moves along the trajectory, shown in heavy black, in this interval by an amount $\Delta t$ (red) and the unit vectors for the momentum change by $\Delta \hat{p} = \hat{p}_f - \hat{p}_i$. Using the fact that the momentum unit vector does not change in magnitude (it can’t since it is a unit vector) then if we find a circle that approximates the local trajectory we can construct similar triangles as shown below.

From the similar triangles, $\frac{|\Delta \hat{p}|}{1} = \frac{v \Delta t}{R}$ or $\frac{|\Delta \hat{p}|}{\Delta t} = \frac{|v|}{R}$. So that describes the change in magnitude of $\frac{d\hat{p}}{dt}$, but in what direction? In order for a vector to change its direction and not its length, the change must take place in a perpendicular direction. In this case it is in the inward radial direction of the osculatory, $\hat{n}$. The limit as $\Delta t$ goes to zero shows that $\frac{d\hat{p}}{dt} = \frac{|v|}{R} \hat{n}$.

Substituting into the expressions for the change in the momentum 

$$\vec{F}_\parallel = \frac{d \hat{p}}{dt} \hat{p} \quad \text{and} \quad \vec{F}_\perp = |\hat{p}| \frac{d\hat{p}}{dt},$$

we can now use the nonrelativistic momentum to find,

$$\vec{F}_\parallel = m \frac{d|v|}{dt} \hat{p} \quad \text{(linear motion) and} \quad \vec{F}_\perp = |\hat{p}| \frac{d\hat{p}}{dt} = m|v| \frac{|v|}{R} \hat{n} = m \frac{v^2}{R} \hat{n} \quad \text{or}$$

$$\vec{F}_\perp = m \frac{v^2}{R} \hat{n} \quad \text{(curvature), where R is the radius of the osculatory circle and } \hat{n} \text{ the radially inward unit vector directed normal to the trajectory or equivalently to the center of the osculatory.}$$
The radius \( R \) of the osculatory is the local radius of curvature of the trajectory. If the orbital equation is known then the radius of curvature can be found from (you will learn this in Calc III):

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R = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} \quad \text{or} \quad R = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}, \quad \text{where} \quad x = x(t) \text{ and } y = y(t)
\]
So now we have the condition for motion in a closed elliptical orbit, \( \vec{F}_\perp = m \frac{v^2}{R} \). This is a very important expression to understand. In order for the object to be traveling in a closed elliptical orbit the component of the net force on the object perpendicular to the trajectory must be equal to \( m \frac{v^2}{R} \).

There is an important special case that comes up often, that of a circular orbit. This is a limiting case of an ellipse with zero eccentricity with both foci at the center of the circle. In this case the trajectory and the osculatory are one in the same. The local curvature of a circle is always its radius, so \( R \) is the radius of the trajectory.
Since $\vec{F}_\parallel = m \frac{d\vec{v}}{dt} \hat{p} = 0$, the speed of the comet is constant. The speed is determined by

$$\vec{F} = \vec{F}_\perp = m \frac{v^2}{R} \hat{n},$$

the radially inward force sets a relationship that must be followed by the comet to move in a circular orbit. Of course, we know what the force due to gravity is between these masses, $\vec{F}_{c,S} = G \frac{Mm}{R^2} \hat{n}$ equating this with the force needed to keep a circular orbit yields,

$$m \frac{v^2}{R} \hat{n} = G \frac{Mm}{R^2} \hat{n} \Rightarrow v = \sqrt{\frac{GM}{R}}.$$  Thus the speed of an object in circular motion due to gravitational attraction is set by the ratio of the mass of the stationary center to the radius of the orbit. Note this is only true for circular orbits.

Changing topics and leaving the comet, let’s consider a constant mass rocket that is launched into space off the moon but does not escape and falls back to the surface. The moon has no atmosphere, so we don’t need to worry about air resistance. What type of trajectory should the rocket have? Our high school physics prepared you well to say a parabolic trajectory, no air resistance right? Not necessarily. If this is a closed orbit it must be an elliptical trajectory. Look at the diagram below,

![Diagram](image)

We know from the shell theorem that the gravitational force will act on the rocket as if all the mass of the moon is concentrated at its center, figure on the left. This is clearly an elliptical orbit for all the reasons that we have discussed above. The shape of the ellipse is determined by the initial momentum of the rocket and the mass and radius of the moon.

Of course, the moon is not really a point particle; it only acts that way for masses above its surface. When the rocket comes around it crashes back into the surface, as in the right hand figure.
So the trajectory can be elliptical. Then why might you think that it should be parabolic? Because in high school you learned that the force of gravity was uniform, giving a parabolic path, and did not consider trajectories where the height was a significant fraction of the size of the massive object. In fact, as the rocket moves away from the moon the gravitational force decreases. If you make the approximation that gravity is a constant force $F = mg$ then your solution is parabolic, but this is a poor approximation for long or high trajectories.
So let’s put some of this theory to work. You are traveling on in a spaceship to mars through the asteroid belt. You need to stop to make some repairs and spot a convenient asteroid, R2D2. From your great distance you can only measure its rough diameter 121400. The density of this type of rock is approximately 5000 kg/m$^3$. Before landing you need to get in a circular orbit with a radius of 120 km. What is the speed that you need to have in this circular orbit?

As you draw closer to R2D2 you realize that your spherical approximation is very crude; with better resolution you determine that R2D2 actually looks like this:

How is the irregular shape of the asteroid going to affect your orbit?

You want to simulate what your orbit will be around R2D2. How will you go about determining the gravitational attraction that you will feel at each point? Write down the algorithm (procedure) to calculate this.