force and dynamics with a spring, analytic approach

It may strike you as strange that the first force we will discuss will be that of a spring. It is not one of the four universal forces and we don’t use springs every day. Or do we? The basic motion of a spring is to oscillate. Many, many mechanical systems oscillate. However, they might not appear to be springs. Take a tree branch bouncing up and down in the wind, is it a spring? How about the vibrations between two atoms in a molecule or a cork bobbing up and down on the surface of water, are these spring systems? From a physics point of view, all of these can be modeled by the same equations as those that describe a spring. This is why examining spring systems is so important. Later in the class we will make this mathematically precise, but for now just remember that the study of springs is far, far more general and important than a coil of metal with a mass hanging from it.

The force law for an ideal spring is \( \vec{F} = -k(\vec{L} - \vec{L}_0) = -k\Delta\vec{L} \). This says that the force that a spring applies is proportional to the amount that the spring is stretched from its equilibrium length. The constant of proportionality is negative, meaning that the force is directed in the opposite direction from the extensive or compressive displacement \( \Delta L \). Let’s look at the simplest spring system:

In this figure the mass is displaced to the right and the spring exerts a restoring force to the left.

Writing Newton’s equation for motion in the lateral direction gives \( \frac{dP}{dt} = -k(x - L_0) \),

combining with \( \frac{dx(t)}{dt} = v(t) \) and nonrelativistically \( p(t) = mv(t) \) gives \( \frac{d^2x(t)}{dt^2} = -\frac{k}{m}(x - L_0) \).

This is a linear second-order inhomogeneous differential equation with constant coefficients (that is a mouth-full), with a harmonic solution.

We have turned the physical problem of a spring, through Newton’s equations of motion, into a mathematical problem: solving a differential equation. Now we need to solve the equation to learn about the behavior of the mass-spring system. There are a number of ways to solve a differential equation; one is simply guessing the solution. This is perfectly acceptable, but often difficult. The equation before us, \( \frac{d^2x(t)}{dt^2} = -\frac{k}{m}(x - L_0) \), is quite simple. We can guess one solution right away, \( x(t) = L_0 \). This is a correct solution, however pretty boring. The mass is
simply sitting at the equilibrium position of the spring and not moving. To find more interesting solutions we need to consider the equation that removes that particular solution:

\[
\frac{d^2 x(t)}{dt^2} = -\frac{k}{m} x(t) .
\]

This equation says that after taking two derivatives of the function you get the function back multiplied by a constant. This leads us to consider a function that when differentiated remains proportional to itself: namely the exponential \( x(t) = A e^{\omega t} \). Rewriting the equation as

\[
\frac{d^2 x(t)}{dt^2} + \frac{k}{m} x(t) = 0
\]

and substituting in the exponential gives \( \alpha^2 + \frac{k}{m} = 0 \). This is called the secular equation. The solutions are \( \alpha = \pm \sqrt{-\frac{k}{m}} \). Now, this is an interesting situation because both \( k \) and \( m \) are positive, but let’s proceed. So we have two independent solutions for the problem \( x_1(t) = A e^{i\sqrt{k/m} t} \) and \( x_2(t) = A e^{-i\sqrt{k/m} t} \). The general solution is the sum of these, plus the particular solution \( x(t) = L_0 \). Specifically the general solution is \( x(t) = A e^{i\sqrt{k/m} t} + A e^{-i\sqrt{k/m} t} + L_0 \). This equation has two adjustable constants \( A_1 \) and \( A_2 \) . These can be determined by using the initial conditions.

So, we have analytically solved the mathematical problem. The only issue is that the solution is complex and certainly not the simple oscillatory motion that we expected. Let’s start by renaming the quantity \( \frac{k}{m} \equiv \omega \). \( \omega \) has the units of inverse time and is called the angular frequency. We need to understand what the complex exponential \( e^{i\omega t} \) means. The best way to do this is expand the function in a Taylor series.

\[
e^{i\omega t} = 1 + i\omega t - \frac{\omega^2}{2!} t^2 - i \frac{\omega^3}{3!} t^3 + \frac{\omega^4}{4!} t^4 + i \frac{\omega^5}{5!} t^5 - \frac{\omega^6}{6!} t^6 - i \frac{\omega^7}{7!} t^7 + \frac{\omega^8}{8!} t^8 + ... \]

This infinite series does not look much better. It is still complex and contain an infinite number of terms. Let’s simplify a bit by collecting real and imaginary terms:

\[
e^{i\omega t} = \left( 1 - \frac{\omega^2}{2!} t^2 + \frac{\omega^4}{4!} t^4 - \frac{\omega^6}{6!} t^6 + \frac{\omega^8}{8!} t^8 + ... \right) + i \left( \omega t - \frac{\omega^3}{3!} t^3 + \frac{\omega^5}{5!} t^5 - \frac{\omega^7}{7!} t^7 + ... \right)
\]

or

\[
e^{i\omega t} = \sum_{n=0}^{\infty} \frac{(\omega t)^{2n}}{2n!} + i \sum_{n=0}^{\infty} \frac{(\omega t)^{2n+1}}{(2n+1)!}
\]
Again, this might not seem to improve the situation until you realize that those two summations are the Taylor series expansions for the cosine and sine function. Go ahead and try it yourself, expand the cosine and sine function and find that

\[
\cos(\omega t) = 1 - \frac{\omega^2}{2!} t^2 + \frac{\omega^4}{4!} t^4 - \frac{\omega^6}{6!} t^6 + \frac{\omega^8}{8!} t^8 + \ldots = \sum_{n=0}^{\infty} \frac{(\omega t)^{2n}}{2n!}
\]

and

\[
\sin(\omega t) = \omega t - \frac{\omega^3}{3!} t^3 + \frac{\omega^5}{5!} t^5 - \frac{\omega^7}{7!} t^7 + \ldots = \sum_{n=0}^{\infty} \frac{(\omega t)^{2n+1}}{(2n+1)!}.
\]

This simplifies things a bit and produces one of the most amazing equations in mathematics, namely Euler's relation: \( e^{i\omega t} = \cos(\omega t) + i \sin(\omega t) \). Now this is progress: we have produced the common oscillatory functions that we expected. However, it is still a complex expression. That can be fixed. Look at the similar expression for the other solution \( e^{-i\omega t} = \cos(\omega t) - i \sin(\omega t) \), here we have used the fact that cosine is an even function, and sine is odd. By adding these two solutions together we get \( \cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2} \). This is still a solution to the original problem.

Similarly subtraction yields \( \sin(\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2i} \). Having now done all of this we have reformed our original complex solution \( x(t) = A e^{i \frac{k}{\sqrt{m}} t} + A e^{-i \frac{k}{\sqrt{m}} t} + L_0 \) into a new form

\[
x(t) = A \sin(\sqrt{\frac{k}{m}} t) + B \cos(\sqrt{\frac{k}{m}} t) + L_0
\]

that contain only real functions (\( A \) and \( B \) could still be complex if we wanted, typically we don’t).

Often the solution is written in a second form that contains a phase angle. This change is simply a trigonometric identity. \( x(t) = A \sin(\sqrt{\frac{k}{m}} t) + B \cos(\sqrt{\frac{k}{m}} t) + L_0 \) or \( x(t) = C \sin(\sqrt{\frac{k}{m}} t + \varphi) + L_0 \)

These two expressions for \( x(t) \) are equivalent in that there are two free parameters in each \( A, B \) in the first and \( C, \varphi \) in the second and the function is periodic with frequency \( f = \frac{1}{2\pi \sqrt{\frac{k}{m}}} \). The two free parameters can be determined from two additional conditions such as the initial position and initial velocity.
There is a relationship between the parameters in each expression so that you can go from one to the other as suits your needs:

\[ C = \sqrt{A^2 + B^2} \quad \text{and} \quad \varphi = \tan^{-1}\left(\frac{B}{A}\right), \]

alternately \( A = C \cos \varphi \) and \( B = C \sin \varphi \). This can be remembered from basic trigonometry, see the figure.

![Diagram](image)

For now we will use \( x(t) = C \cos\left(\frac{k}{\sqrt{m}} t + \varphi\right) + L_0 \)

The constant \( L_0 \) is a reflection of where the origin is placed. If it is placed at the equilibrium location of the end of the spring, \( L_0 \) will be zero. This is a common way to set up your coordinate system. The amplitude of oscillation is given by \( C \), and the range of \( y \) varies from \(-C\) to \(+C\). The term inside the parentheses, \( \frac{k}{\sqrt{m}} t + \varphi \), is called the phase. There is a time independent term \( \varphi \), called the phase shift, initial phase, or phase constant, that determines where in the cycle the motion starts at time zero. In other words, it determines whether the object starts at the equilibrium position and has a nonzero momentum or at a point of maximum displacement with zero momentum, or anywhere in between these positions. To determine \( C \) and \( \varphi \) you must be given two pieces of information about the system, such as from what position did it start and how fast was it going initially. This will allow you to uniquely determine these constants.

So what about the time dependent part of the phase, \( \frac{k}{\sqrt{m}} t \)? This term determines the frequency of oscillation. Every time this part of the phase increases by \( 2\pi \) the motion has completed one
more full cycle. That means that the period \( T \) for an oscillation is given by \( \sqrt{\frac{k}{m}} = 2\pi \) or

\[
T = 2\pi \sqrt{\frac{m}{k}}.
\]

If that is the time per cycle, the number of cycles per second (Hz), the frequency \( f \), can be found from \( f = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \). A closely related quantity, the angular frequency \( \omega \), gives the oscillation rate in radians per second and is defined by \( \omega = 2\pi f = \sqrt{\frac{k}{m}} \).

Much of the behavior of the oscillator can be understood from this figure:

![Diagram of harmonic motion](image)

where \( \theta(t) = \omega t + \phi \). Picture a vector rotating with a constant rate, i.e. the second hand on a clock, only this rotates counter-clockwise. The position of the oscillator is given by the x-component of the vector, i.e. where the dashed line on the figure intersects the x-axis. The vector rotates at a rate \( \omega \) radians per second and the position of the oscillator ranges from \( C \) to \( -C \). This is called harmonic motion. On a position vs. time plot the motion looks like a “sine” curve:
The parameters that were used are \( C = 1, \omega = 1, \varphi = \frac{3\pi}{4} \).

The program at the end of this section has an animation with a rotating vector and the position of the y-coordinate tracing out oscillations of harmonic motion.
# Plots sine curve and rotating vector representation for harmonic motion

from __future__ import division
from visual import *
from visual.graph import * # import graphing features

sine_function = gdisplay( title = 'harmonic position vs. time',
                          xtitle = 'time',
                          ytitle = 'position',
                          x=0, y=400, width=1000,
                          foreground=color.black,
                          background=color.white) #set display #1

sine_func = gdots(gdisplay = sine_function, color = color.black)  # curve for sine

dt = .1  #graphing increment
C = 1    # C is amplitude of oscillation
w = 1    # angular frequency
phi = 3*3.14/4  # phase shift, set initial conditions

#graphics
vector = arrow(pos=(0,0,0), axis=(C*cos(phi),C*sin(phi),0), shaftwidth=.01)
position = sphere(pos = (C*cos(phi),0,0), radius = .05, color = color.red)

for t in arange (0,  20,  dt):                          # time points from 0 to 10 interval dt
    rate(10)
    sine = C*cos(w*t + phi)                             # harmonic function
    sine_func.plot (pos=(t, sine ))                     # plot harmonic function
    vector.axis = (C*cos(w*t + phi),C*sin(w*t + phi),0) # rotating arrow
    position.pos = (C*cos(w*t + phi),0,0)              # x position of arrow