Problem Set # 1: The Continuous-Time Bellman Equation for General Recursive Utility

Economics 609, Winter 2003

Due: Tuesday, January 21, 2:30

Consider the case of preferences given by the general form

\[ v_t = \Psi_t(K_t, X_t, E_t v_{t+h}; h). \]

When \( h = 0 \), the function \( \Psi \) satisfies

\[ \Psi_t(K_t, X_t, v_t; 0) = v_t \]

Also, the function \( \Psi \) has continuous first derivatives. The direct partial derivative with respect to the last argument \( h \) after the semicolon has the limit as \( h \to 0 \) of

\[ \Psi_h(K_t, X_t, v_t; 0) = G(K_t, X_t, v_t). \]

The contemporaneous constraint is of standard form:

\[ X_t \in X^t(K_t). \]

The transition equation for \( K_t \) (=intertemporal budget constraint) is given by

\[ K_{t+h} = K_t + hA^t(K_t, X_t) + \sqrt{h}\Omega^t(K_t, X_t)\epsilon_{t+h} \]

where \( \epsilon_{t+h} \) is an i.i.d. binomial random variable equal to +1 with probability .5 and equal to -1 with probability .5.

1. Write down the recursion equation for the value function \( V^t(K_t) \) in discrete time with time interval \( h \).

2. The continuous-time value function \( V^t(K_t) \) satisfies the limit of what this recursion equation becomes when \( h \to 0 \). Take this limit, using L’Hopital’s rule where appropriate. Assume that \( V \) has continuous second derivatives and find a partial differential equation for \( V^t(K_t) \). This partial differential equation will have maximization in the middle of it. Show your derivation.

Hint: Treat \( V^{t+h}(K_t) \) as a function of \( h \) only through the time superscript \( t + h \). The justification for this is that we are solving for the continuous-time differential equation that applies to whatever the continuous-time limit of \( V^t(K_t) \) is. So \( V^t(K_t) \) is like the unknown we are trying to solve for. And we are only trying to solve for the limit of \( V^t(K_t) \) when \( h \to 0 \). In any case, you will have a hard time doing the problem if you try to treat \( V^t(K_t) \) as a function of \( h \), or \( V^{t+h}(K_t) \) as a function of \( h \) other than through the time superscript \( t + h \).