Consider two facts.

**Fact A:** The papers “Risk Aversion with Random Initial Wealth” by Richard Kihlstrom, David Romer and Steve Williams (*Econometrica*, 49 (1981), 911–920) and “Preservation of ‘More Risk Averse’ Under Expectations,” by D. C. Nachman (*Journal of Economic Theory*, 23 (1982), 361–368), show that if one function $u_2(c)$ is globally more risk aversion than another function $u_1(c)$ that as long as either of the two functions has decreasing absolute risk aversion, adding the same independent random variable to the arguments of both functions and taking expectations over that ‘background’ random variable will yield functions that have the same global risk aversion ranking:

$\hat{u}_2(c)$ is globally more risk averse than $\hat{u}_1(c)$, where

$$\hat{u}_i(c) = E \epsilon u_i(c + \tilde{\epsilon}).$$

**Fact B:** The paper “Standard Risk Aversion” by Miles Kimball (*Econometrica* 61 (1993), 589–611), which is in the coursepack, the prior papers “Proper Risk Aversion,” by John Pratt and Richard Zeckhauser (*Econometrica* 55 (1987), 143–154) and subsequent paper “Risk Vulnerability and the Tempering Effect of Background Risk,” by Christian Gollier and John Pratt (*Econometrica* 64 (1996), 1109–1124) each imply among other things that if $u_1$ has constant relative risk aversion of $\gamma$ and $\epsilon$ is a mean-zero background risk, then for all $c$,

$$\frac{-cu_1''(c)}{u_1'(c)} \geq \gamma$$

1. Use Facts A and B in conjunction to argue that

$$\frac{-cu_2''(c)}{u_2'(c)} \geq \gamma$$

(Proof) Suppose $u_1(c)$ has constant relative risk aversion of $\gamma$ and $u_2(c)$ is globally more risk averse than $u_1(c)$, i.e, for all $c$,

$$\frac{-cu_1''(c)}{u_1'(c)} = \gamma \quad \text{and} \quad \frac{-cu_2''(c)}{u_2'(c)} \geq \gamma.$$

Then the function $u_1(c)$ has decreasing absolute risk aversion since

$$\frac{d}{dc} \left( \frac{-u_1''(c)}{u_1'(c)} \right) = \frac{d}{dc} \left( \frac{\gamma}{c} \right) = -\frac{\gamma}{c^2} \leq 0.$$

Let’s say that $\tilde{\epsilon}$ is a mean-zero background risk as well as an independent random variable. By Fact B, for all $c$,

$$\frac{-cu_1''(c)}{u_1'(c)} \geq \gamma.$$

Since $\hat{u}_2(c)$ is globally more risk averse than $\hat{u}_1(c)$ by Fact A, we have for all $c$,

$$\frac{-cu_2''(c)}{u_2'(c)} \geq \frac{-cu_1''(c)}{u_1'(c)} \geq \gamma.$$
Thus,
\[-cu''_2(c) = \frac{\gamma}{u''_2(c)} \leq -cu''_2(c) = \frac{\gamma}{u''_2(c)} \geq \gamma.\]

2. Show that the property \( P \) defined by for all \( \theta \in [0, 1] \) and \( B_1, B_2 \geq 0 \),
\[
V \left( \left( \theta B_1^{1-\gamma} + (1 - \theta)B_2^{1-\gamma} \right)^{\frac{1}{1-\gamma}} \right) - \theta V(B_1) - (1 - \theta)V(B_2) \geq 0
\]
is equivalent to the relative risk aversion of \( V \) being greater than or equal to \( \gamma \) whenever the relevant derivatives exist.

(Proof) The assumption that the relative risk aversion of \( V \) is greater than or equal to \( \gamma \) is true if and only if \( V(B) \equiv \frac{\theta B^{1-\gamma}}{1-\gamma} \). It means that there exists an increasing and concave function \( \varphi \) such that
\[
V(B) = \varphi(u(B)).
\]
Since the function \( \varphi \) is increasing and concave, for all \( \theta \in [0, 1] \) and \( \delta \in \mathbb{R} \),
\[
\varphi (\theta u_1 + (1 - \theta)u_2) - \theta \varphi(u_1) - (1 - \theta)\varphi(u_2) \geq 0,
\]
where \( u_1 = u(B_1) \) and \( u_2 = u(B_2) \).

The left-hand side is:
\[
\varphi (\theta u_1 + (1 - \theta)u_2) - \theta \varphi(u_1) - (1 - \theta)\varphi(u_2) = V \left( u^{-1}(\theta u_1 + (1 - \theta)u_2) \right) - \theta V(B_1) - (1 - \theta)V(B_2)
\]
\[
= V \left( ((1 - \gamma) \theta u_1 + (1 - \theta)u_2)^{\frac{1}{1-\gamma}} \right) - \theta V(B_1) - (1 - \theta)V(B_2)
\]
\[
= V \left( \theta B_1^{1-\gamma} + (1 - \theta)B_2^{1-\gamma} \right)^{\frac{1}{1-\gamma}} - \theta V(B_1) - (1 - \theta)V(B_2).
\]
Thus
\[
V \left( \theta B_1^{1-\gamma} + (1 - \theta)B_2^{1-\gamma} \right)^{\frac{1}{1-\gamma}} - \theta V(B_1) - (1 - \theta)V(B_2) \geq 0.
\]

3. Consider the consumption-savings problem with the Bellman equation
\[
V^t(B_t) = \max_{C_t} \frac{C_t^{1-\gamma}}{1-\gamma} + e^{-\rho h} E_t V^{t+h}(R_t(B_t - C_t) + \epsilon^t(\omega_{t+h})),
\]
where the functions \( \epsilon^t \) are known in advance, but the realizations of \( \omega_{t+h} \) are not. The random variable \( \epsilon \) has mean zero. Assuming \( V^{T+h}(B_{T+h}) = 0 \), show that \( V^t(B_t) \) has the property \( P \):
\[
V \left( \theta B_1^{1-\gamma} + (1 - \theta)B_2^{1-\gamma} \right)^{\frac{1}{1-\gamma}} - \theta V(B_1) - (1 - \theta)V(B_2) \geq 0.
\]
(Proof) Let’s use backward induction here.
(step 1) Show that $V^T$ has the property $P$. 

Since $V^{T+h}(B_{T+h}) \equiv 0$, there is no incentive to save at $T$ so that $C_T = B_T$. Thus at the last period $T$, 

$$V^T(B_T) = \frac{B_T^{1-\gamma}}{1-\gamma},$$

Hence, $V^T$ has constant relative risk aversion of $\gamma$, that is, the property $P$ holds for $V^T$ with equality.

Through the following two steps, we will show that the property $P$ holds for $V^t$ if the property $P$ holds for $V^{t+h}$.

(step 2) Show that the property $P$ is preserved under expectation and then the unmaximized value $F$ has the appropriate property $Q$.

First, define $S \equiv (B-C)$. Since $V^{t+h}(B)$ has the property $P$ for all $B > 0$ and $R_tS_t > 0$, $V^{t+h}(R_tS_t)$ has the property $P$. Since 

$$V^{t+h} \left((\theta(RS_1)^{1-\gamma} + (1-\theta)(RS_2)^{1-\gamma})^{1/1-\gamma}\right) - \theta V^{t+h}(RS_1) - (1-\theta)V^{t+h}(RS_2) \geq 0$$

$$\iff V^{t+h} \left(\theta S_1^{1-\gamma} + (1-\theta)S_2^{1-\gamma}\right)^{1/1-\gamma} - \theta V^{t+h}(RS_1) - (1-\theta)V^{t+h}(RS_2) \geq 0,$$

$V^{t+h}$ has the property $P$ in $S_t$. Using Facts A and B as in part 1, we can say that the relative risk aversion of $V^{t+h}(S_t) \equiv E_tV^{t+h}(R_tS_t + \epsilon^t)$ is greater than or equal to $\gamma$. Thus the property is preserved under expectation.

Second, define the unmaximized value function:

$$F^t(B_t, S_t) = \frac{(B_t - S_t)^{1-\gamma}}{1-\gamma} + e^{-\rho h} E_t V^{t+h}(R_tS_t + \epsilon^t(\omega_{t+h}))$$

and then prove that $F^t(B_t, S_t)$ has the associated property $Q$ in $(B_t, S_t)$, where the property $Q$ is defined by for all $\theta \in [0, 1]$ and $(B_1, S_1)$ and $(B_2, S_2) \in \mathbb{R}_+^2$,

$$F \left(\left(\theta B_1^{1-\gamma} + (1-\theta)B_2^{1-\gamma}\right)^{\gamma}ight) - \theta F(B_1, S_1) - (1-\theta)F(B_2, S_2) \geq 0.$$

By the facts that the sum of several utility functions that each has relative risk aversion greater than $\gamma$ has relative risk aversion greater than $\gamma$ and the property is preserved under expectation, it suffices to show that $\frac{(B_t - S_t)^{1-\gamma}}{1-\gamma}$ has the property $Q$. By Minkowski’s Inequality,

$$u \left(\left(\theta B_1^{1-\gamma} + (1-\theta)B_2^{1-\gamma}\right)^{\gamma}\right) - \theta u(B_1 - S_1) - (1-\theta)u(B_2 - S_2) \geq u(c)^{\gamma}$$

where $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$.

Thus the property $Q$ holds for $F$.

(Step 3) Let’s apply the Preser-Max theorem.
Define $S_1 \in \text{arg} \max_{S \in [0, B_1]} F^t(B_1, S)$, $S_2 \in \text{arg} \max_{S \in [0, B_2]} F^t(B_2, S)$ and $(B_3, S_3) = \left( \left( \theta B_1^{1-\gamma} + (1 - \theta) B_2^{1-\gamma} \right)^{\frac{1}{1-\gamma}}, \left( \theta S_1^{1-\gamma} + (1 - \theta) S_2^{1-\gamma} \right)^{\frac{1}{1-\gamma}} \right)$, where $S_3 \in [0, B_3]$ by definition.

For all $\theta \in [0, 1]$,

$$V^t \left( \left( \theta B_1^{1-\gamma} + (1 - \theta) B_2^{1-\gamma} \right)^{\frac{1}{1-\gamma}} \right) - \theta V^t(B_1) - (1 - \theta) V^t(B_2) = \max_{S \in [0, B_2]} F^t \left( \left( \theta B_1^{1-\gamma} + (1 - \theta) B_2^{1-\gamma} \right)^{\frac{1}{1-\gamma}}, S \right) - \theta \max_{S \in [0, B_1]} F^t(B_1, S) - (1 - \theta) \max_{S \in [0, B_2]} F^t(B_2, S) \geq F^t \left( \left( \theta B_1^{1-\gamma} + (1 - \theta) B_2^{1-\gamma} \right)^{\frac{1}{1-\gamma}}, \left( \theta S_1^{1-\gamma} + (1 - \theta) S_2^{1-\gamma} \right)^{\frac{1}{1-\gamma}} \right) - \theta F^t(B_1, S_1) - (1 - \theta) F^t(B_2, S_2) \geq 0$$

The last inequality holds because the function $F$ has the property $Q$.

(Step 4) By recursion, we can say that $V^{T-nh}(B)$ has the property $P$ for all $n \in \{0, 1, 2, \ldots \}$.

4. Explain why the result of question 3 carries over to the value function $V^t(B_t, Z_t)$ where $Z$ is a vector of exogenous state variables, with the gross real interest rate $R$ a function $R^t(Z_t)$ of this vector, $\epsilon$ a function of this vector $\epsilon^t(Z_t, \omega_{t+h})$ and the vector of exogenous state variables itself evolving according to $Z_{t+h} = \Gamma^t(Z_t, \omega_{t+h}; h)$.

(Explanation) Adding a vector exogenous state variables $Z$ to the model will affect only (step 2) of the above procedure. If the realization of the gross real interest rate $R^t(Z_t)$ is always positive, $V^{t+h}(R^t S_t)$ still has the property $P$ in $S_t$. And also the facts that the realization of $R^t(\Gamma^t(Z_{t-h}, \omega_t; h))$ and $\epsilon^t(\Gamma^t(Z_{t-h}, \omega_t; h), \omega_{t+h})$ are independent of each other and $\epsilon^t$ has a zero mean imply that the property $P$ is preserved under expectation by Facts A and B.

5. Modifying the problem in question 3, let there be a two-point risky-asset and risk-free asset that allow one to construct an optimist’s security that pays 1 unit if the two-point risky asset does well and zero otherwise and a pessimist’s security that pays 1 unit if the two-point risky asset does badly and zero otherwise. The gross return on the optimist’s security is $R_t^+$ or 0, while the gross return of the pessimist’s security is 0 or $R_t^-$ if the two-point risky asset does well or badly. The probability of the good outcome for the risky asset is $p$. This gives rise to the Bellman equation

$$V^t(B_t) = \max_{S_t^+, S_t^-} \frac{(B_t - S_t^+ - S_t^-)^{1-\gamma}}{1-\gamma}$$

$$+ p e^{-ph} E_t V^{t+h}(R_t^+ S_t^+ + \epsilon^t(\omega_{t+h}))$$

$$+ (1 - p) e^{-ph} E_t V^{t+h}(R_t^- S_t^- + \epsilon^t(\omega_{t+h}))$$

where the expectation $E_t$ only applies to $\epsilon$. Prove that $V$ has the property $P$ above.
Let’s follow the above procedure.

(Step 1) Obviously it is the same as in (step 1) of part 3. At the last period, $V^T(B_T) = \frac{B_T^{1-\gamma}}{1-\gamma}$. Hence, $V^T$ has constant relative risk aversion of $\gamma$, that is, the property $P$ holds for $V^T$ with equality.

(Step 2) Suppose $V^{t+h}$ has the property $P$. Then we can prove that $V^{t+h}(R^+_t S^+_t)$ and $V^{t+h}(R^-_t S^-_t)$ have the property $P$ in $S^+_t$ and $S^-_t$, respectively.

$$V^{t+h} \left( R^+ (\theta(S^+_1)^{1-\gamma} + (1-\theta)(S^+_2)^{1-\gamma}) \right) = \theta V^{t+h}(R^+_1 S^+_1) - (1-\theta)V^{t+h}(R^+_2 S^+_2)$$

By the same way, we can show that $V^{t+h}(R^-_1 S^-_1)$ has the property $P$ in $S^-_t$.

Now let’s check if the property $P$ is preserved under expectation. Applying Facts A and B to each expectation in the Bellman equation yields that both $E_tV^{t+h}(R^+_t S^+_t + \epsilon'(\omega_{t+h}))$ and $E_tV^{t+h}(R^-_t S^-_t + \epsilon'(\omega_{t+h}))$ have the property $P$. With the fact that the sum of several utility functions that each has relative risk aversion greater than $\gamma$ has relative risk aversion greater than $\gamma$, we can tell that $p e^{-\rho h}E_tV^{t+h}(R^+_t S^+_t + \epsilon'(\omega_{t+h})) + (1-p)e^{-\rho h}E_tV^{t+h}(R^-_t S^-_t + \epsilon'(\omega_{t+h}))$ has the property $P$. Thus the property is preserved under expectation.

Define the unmaximized value function $F^t(B, S^+, S^-)$:

$$F^t(B, S^+, S^-) = \frac{(B_t - S^+_t - S^-_t)^{1-\gamma}}{1-\gamma} + p e^{-\rho h}E_tV^{t+h}(R^+_t S^+_t + \epsilon'(\omega_{t+h})) + (1-p)e^{-\rho h}E_tV^{t+h}(R^-_t S^-_t + \epsilon'(\omega_{t+h}))$$

and then prove that $F^t(B, S^+, S^-)$ has the associated property $Q$ in $(B, S^+, S^-)$, where the property $Q$ is defined by for all $\theta \in [0, 1]$ and $(B_1, S^+_1, S^-_1)$ and $(B_2, S^+_2, S^-_2) \in \mathbb{R}^3$:

$$F \left( \left( \theta B_1^{1-\gamma} + (1-\theta)B_2^{1-\gamma} \right) \right) - \theta F(B_1, S^+_1, S^-_1) - (1-\theta)F(B_2, S^+_2, S^-_2) \geq 0.$$  

By the facts that the sum of several utility functions that each has relative risk aversion greater than $\gamma$ has relative risk aversion greater than $\gamma$ and the property $P$ is preserved under expectation, it suffices to show that $u(B_t - S^+_t - S^-_t) = \frac{(B_t - S^+_t - S^-_t)^{1-\gamma}}{1-\gamma}$ has the property $Q$ in $(B, S^+, S^-)$.

Let’s define $\tilde{B} \equiv \left( \theta B_1^{1-\gamma} + (1-\theta)B_2^{1-\gamma} \right)^{1-\gamma}$, $\tilde{S}^+ \equiv \left( \theta(S^+_1)^{1-\gamma} + (1-\theta)(S^+_2)^{1-\gamma} \right)^{1-\gamma}$, and $\tilde{S}^- \equiv \left( \theta(S^-_1)^{1-\gamma} + (1-\theta)(S^-_2)^{1-\gamma} \right)^{1-\gamma}$. Then,

$$u(\tilde{B}, \tilde{S}^+, \tilde{S}^-) - \theta u(B_1, S^+_1, S^-_1) - (1-\theta)u(B_2, S^+_2, S^-_2) = \theta \left( B_1 - S^+_1 - S^-_1 \right)^{1-\gamma} - \theta \left( B_2 - S^+_2 - S^-_2 \right)^{1-\gamma} \geq 0,$$

where the above inequality holds by Minkowski’s Inequality. Hence the associated property $Q$ holds for $F$.  

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(Step 3) Let’s use the Preser-Max theorem.

Define \((S_1^+, S_1^-) \in \arg \max_{S^+, S^- \in [0, B_1]} F^t(B_1, S^+, S^-)\) and \((S_2^+, S_2^-) \in \arg \max_{S^+, S^- \in [0, B_2]} F^t(B_2, S^+, S^-)\).

For all \(\theta \in [0, 1],\)

\[
V^t(\hat{B}) - \theta V^t(B_1) - (1 - \theta) V^t(B_2) = \max_{S^+, S^- \in [0, B]} F^t(\hat{B}, S^+, S^-) - \theta \max_{S^+, S^- \in [0, B_1]} F(B_1, S^+, S^-) - (1 - \theta) \max_{S^+, S^- \in [0, B_2]} F(B_2, S^+, S^-) \\
\geq F^t(\hat{B}, \hat{S}^+, \hat{S}^-) - \theta F(B_1, S_1^+, S_1^-) - (1 - \theta) F(B_2, S_2^+, S_2^-) \\
\geq 0.
\]

The last inequality holds because the function \(F\) has the property \(Q\) in \((B, S^+, S^-)\). I will skip the feasibility proof that \(\hat{S}^+ + \hat{S}^- \leq \hat{B}\). (For the proof, refer to the solution for the take-home exam #3, 2a in Winter 2001.)

(Step 4) By recursion, we can say that \(V^t\) has the property \(P\) for all \(t \leq T\).

6. Discuss ways in which the result to question 5 can be generalized—especially along the lines of question 4. One of the harder aspects of this question is whether the result carries over when there is a separate mean-zero \(\epsilon^+\) function and \(\epsilon^-\) function.

We can add a vector of exogenous variables \(Z\) as in part 4. As long as the realization of \(R^t(Z_t)\) is independent of a mean-zero background risk \(\epsilon_t\), the result of question 5 carries over to the value function \(V^t(B_t, Z_t)\). Even though there are separate mean-zero \(\epsilon^+\) function and \(\epsilon^-\) function, the property \(P\) is preserved under expectation by Facts of A and B. Thus, \(E_t V^{t+h}(R_t^+ S_t^+ + \epsilon^{t+}(\omega_{t+h}))\) and \(E_t V^{t+h}(R_t^- S_t^- + \epsilon^{-t}(\omega_{t+h}))\) have the property \(P\). The fact that the sum of several utility functions that each has relative risk aversion greater than \(\gamma\) has relative risk aversion greater than \(\gamma\) implies that \(p e^{-\rho h} E_t V^{t+h}(R_t^+ S_t^+ + \epsilon^{t+}(\omega_{t+h})) + (1 - p) e^{-\rho h} E_t V^{t+h}(R_t^- S_t^- + \epsilon^{-t}(\omega_{t+h}))\) still has the property \(P\). Thus, the result of question 5 carries over when there are separate mean-zero \(\epsilon^+\) function and \(\epsilon^-\) function.