1. The Firm Problem

\[ V^t(K_t) = \max_{U,\varepsilon,H,N,M,I} \mathbb{E}_t \sum_{\tau = t+1}^{\infty} \frac{\beta^t}{\beta^\tau} \{ P_{Y,t} F(U_t, K_t, \varepsilon_t, H_t, N_t, M_t, I_t, Z_t) \} \]

s.t.,

\[ K_{t+h} = K_t \Gamma(I_t/K_t, U_t; Z_t), \quad U_t \leq U. \]

a. The recursive form of the Bellman equation:

\[ V^t(K_t) = \max_{U,\varepsilon,H,N,M,I} \{ P_{Y,t} F(U_t, K_t, \varepsilon_t, H_t, N_t, M_t, I_t, Z_t) - W_t G^t(H_t, \varepsilon_t, U) N_t - P_{M,t} M_t - P_{I,t} K_t \Phi(I_t/K_t) \} + E_t \frac{\partial}{\partial K_t} V^{t+h}(K_t \Gamma(I_t/K_t, U_t; Z_t)). \]

b. Consider the following transformation for all \( \alpha \geq 0 \).

\[
\begin{align*}
\text{Transformation(T)} & \quad \text{Pref. Symmetry(S)} \\
K & \rightarrow \alpha K & \nu \rightarrow \alpha \nu \\
N & \rightarrow \alpha N \\
M & \rightarrow \alpha M \\
I & \rightarrow \alpha I \\
\end{align*}
\]

Notice that

1. the Contemporaneous constraint symmetry holds since \( U \) is not transformed.
2. the symmetry for the transition equation since

\[ K_{t+h} = K_t \Gamma(I_t/K_t, U_t; Z_t) \leftrightarrow \alpha K_{t+h} = \alpha K_t \Gamma(\alpha I_t/\alpha K_t, U_t; Z_t) \]

(3) the preference symmetry holds because the function \( F \) has constant returns to scale in \( U, \varepsilon H, N, \) and \( M \):

\[
\begin{align*}
\nu^t & = P_{Y,t} F(U_t, K_t, \varepsilon_t, H_t, N_t, M_t, I_t, Z_t) - W_t G^t(H_t, \varepsilon_t, U) N_t - P_{M,t} M_t - P_{I,t} K_t \Phi(I_t/K_t) + E_t \frac{\partial}{\partial K_t} \nu^{t+h}(K_t \Gamma(I_t/K_t, U_t; Z_t)) \\
\iff \alpha \nu^t & = P_{Y,t} \alpha F(U_t, K_t, \varepsilon_t, H_t, N_t, M_t, I_t, Z_t) - W_t G^t(H_t, \varepsilon_t, U) \alpha N_t - P_{M,t} \alpha M_t - P_{I,t} \alpha K_t \Phi(I_t/K_t) + E_t \frac{\partial}{\partial K_t} \alpha \nu^{t+h}(K_t \Gamma(I_t/K_t, U_t; Z_t)) \\
F \text{ has CRS} \iff \alpha \nu^t & = P_{Y,t} F(U_t, \alpha K_t, \varepsilon_t, H_t, \alpha N_t, \alpha M_t, I_t, Z_t) - W_t G^t(H_t, \varepsilon_t, U) \alpha N_t - P_{M,t} \alpha M_t - P_{I,t} \alpha K_t \Phi(I_t/K_t) + E_t \frac{\partial}{\partial K_t} \alpha \nu^{t+h}(K_t \Gamma(\frac{I_t}{K_t}, U_t; Z_t)) \\
\iff S(K_t, \nu^t) & = P_{Y,t} F(U_t, \alpha K_t, \varepsilon_t, H_t, \alpha N_t, \alpha M_t, I_t, Z_t) - W_t G^t(H_t, \varepsilon_t, U) \alpha N_t - P_{M,t} \alpha M_t - P_{I,t} \alpha K_t \Phi(I_t/K_t) + E_t \frac{\partial}{\partial K_t} S(K_t \Gamma(\frac{I_t}{K_t}, U_t; Z_t), \nu^{t+h}).
\end{align*}
\]
Thus, we can apply the symmetry theorem to this firm problem. The symmetry theorem implies that the value function has constant returns to scale in $K$: for all $\alpha \geq 0$

$$V^t(\alpha K_t) = \alpha V^t(K_t).$$

By setting $\alpha = 1/K$, we can show that average $Q$ is equal to marginal $q$.

$$V^t(1) = \frac{1}{K_t} V^t(K_t) \Rightarrow V^t(K_t) = K_t V^t(1).$$

Taking a differentiation with respect to $K$ gives

$$V^t_K(K_t) = V^t(1).$$

Hence, marginal $q = V^t_K(K_t) = V^t(1) = \frac{1}{K_t} V^t(K_t)$.

Using the Euler’s Theorem, we can also see the same result:

$$V^t_K(K_t) K_t = V^t(K_t) \Rightarrow V^t_K(K_t) = \frac{V^t(K_t)}{K_t}.$$  

When applying the above scale symmetry to this firm problem, prices are considered as parameters. So we need to assume that output and factors markets are competitive to have prices constant. In addition, the assumption that $\Gamma$ and $\Phi$ have constant returns to scale in $K$ and $I$ is also needed.

2. Rate of Time Symmetry

a. To show that the transformation is a symmetry of the constraints, first look at the contemporaneous constraint. Since $X$ was not transformed, this constraint is still the same.

In the case of the transition equation, we have:

$$K^{[n]} = K^{[n+1]} + h[rK^{[n+1]} + f(X^{[n+1]} + K^{[n+1]} \sigma g(X^{[n+1]} \sqrt{h} \epsilon^{[n]})]

\iff \frac{K^{[n]}}{\theta} = K^{[n+1]} + h \frac{rK^{[n+1]} + f(X^{[n+1]} + 1} + \frac{1}{\theta} K^{[n+1]} \sigma g(X^{[n+1]} \sqrt{h} \epsilon^{[n]})

\iff \frac{K^{[n]}}{\theta} = K^{[n+1]} + h \frac{rK^{[n+1]} \theta + f(X^{[n+1]} \theta)}{\theta} + \frac{K^{[n+1]} \sigma g(X^{[n+1]} \sqrt{h} \epsilon^{[n]})}{\theta}

Thus the intertemporal budget constraint still holds after this transformation.

For the preference symmetry, define $\Psi(K^{[n+1]}, X^{[n+1]}, E\nu^{[n]}; h, \rho, r) = hU(X^{[n+1]} + e^{-\rho h E\nu^{[n]}}$.

$$\nu^{[n+1]} = hU(X^{[n+1]} + e^{-\rho h E\nu^{[n]}}$$

$$\nu^{[n+1]} = \frac{h}{\theta} U(X^{[n+1]} + e^{-\rho h E\nu^{[n]}})$$

$$S(\nu^{[n+1]}) = \Psi(\frac{K^{[n+1]}}{\theta}, X^{[n+1]}, E\nu^{[n]}; h/\theta, \theta \rho, \theta r).$$

Thus, the preference symmetry is satisfied.

b. Argue that $V$ is not a direct function of time when $U, r, f, \rho, \sigma$ and $g$ are not having time as an argument. Think about the above discrete-time case. Since $V^{[0]}(K) = 0, V^{[0]}(K, t) = 0$ (in fact $t = T$) and $V^{[0]}(K, t +
\( \alpha = 0 \) (in fact \( t + \alpha = T \)). Suppose for all possible \( K \), \( V^{[n]}(K, t) = V^{[n]}(K, t + \alpha) \), for any \( \alpha \in R \). Since \( U, r, f, \rho, \sigma \) and \( g \) are not having time as an argument, we have:

\[
V^{[n+1]}(K, t) = \max_X hU(X) + e^{-\rho h}E_t V^{[n]} = \max_X hU(X) + e^{-\rho h}E_{t+\alpha} V^{[n]} = V^{[n+1]}(K, t + \alpha)
\]

By the mathematical induction, we can conclude that \( V^{[n]}(K, t) = V^{[n]}(K, t + \alpha) \) for any \( \alpha \in R \).

The limit of the discrete-time problem as \( h \rightarrow 0 \) and \( T \rightarrow \infty \) has the same property, i.e., \( V^{[\infty]}(K, t) = V^{[\infty]}(K, 0) \).

b-2) The symmetry theorem says that for any \( \theta \geq 0 \),

\[
V^t(K_t; \theta \rho, \theta r, \theta \sigma) = \frac{1}{\theta} V^t(K_t; \rho, r, \sigma).
\]

Setting \( \theta = \frac{1}{\rho} \) yields

\[
\rho V^t(K_t; \rho, r, \sigma) = V^t(\rho K_t; 1, \frac{r}{\rho}, \frac{\sigma}{\rho}).
\]

Dividing through by \( \rho \) gives

\[
V^t(K_t; \rho, r, \sigma) = \frac{1}{\rho} V^t(\rho K_t; 1, \frac{r}{\rho}, \frac{\sigma}{\rho}).
\]

From this symmetry theorem, we can reduce the dimension of parameters from three to two.