1. The recursion equation for the value function $V^t(K_t)$:

$$V^t(K_t) = \max_{X_t \in \mathcal{X}^t(K_t)} \Psi(K_t, X_t, E_t V^{t+h}(K_{t+h}); h)$$

subject to:

$$K_{t+h} = \Gamma(K_t + hA^t(K_t, X_t) + \sqrt{h\Omega}(K_t, X_t) \varepsilon_{t+h}).$$

Plugging the intertemporal budget constraint into the unmaximized function $\Psi$ yields the following recursion equation:

$$V^t(K_t) = \max_{X_t \in \mathcal{X}^t(K_t)} \Psi(K_t, X_t, E_t V^{t+h}(\Gamma(K_t + hA^t(K_t, X_t) + \sqrt{h\Omega}(K_t, X_t) \varepsilon_{t+h}); h); h).$$

2. The continuous-time Bellman equation:

First, subtract $V^t(K_t)$ from both sides of the recursion equation, divide by $h$, and take the limit as $h \to 0$.

$$0 = \max_{X_t \in \mathcal{X}^t(K_t)} \lim_{h \to 0} \left[ \Psi(K_t, X_t, E_t V^{t+h}(\Gamma(K_t + hA^t(K_t, X_t) + \sqrt{h\Omega}(K_t, X_t) \varepsilon_{t+h}); h); h) - V^t(K_t) \right]/h$$

Since the denominator and the numerator go to zero as $h \to 0$, we can apply L'Hôpital's rule.

$$0 = \max_{X_t \in \mathcal{X}^t(K_t)} \lim_{h \to 0} \left[ \Psi_t E_t \left[V^{t+h}_K [A^t(K_t, X_t) + \frac{\sqrt{\Omega}}{2\sqrt{h}} \varepsilon_{t+h}] + V^{t+h}_t(K_{t+h}) \right] + \Psi_h \right]$$

where

$$\Psi_t^v = \frac{\partial \Psi(K_t, X_t, E_t V^{t+h}(\Gamma(K_t + hA^t(K_t, X_t) + \sqrt{h\Omega}(K_t, X_t) \varepsilon_{t+h}); h); h)}{\partial V^{t+h}}$$

$$E_t V^{t+h}_K [A^t + \frac{\sqrt{\Omega}}{2\sqrt{h}} \varepsilon_{t+h}] = .5V^{t+h}_K (K_t + hA^t(K_t, X_t) + \sqrt{h\Omega}(K_t, X_t) \varepsilon_{t+h}) [A^t + \frac{\sqrt{\Omega}}{2\sqrt{h}}] + .5V^{t+h}_K (K_t + hA^t(K_t, X_t) - \sqrt{h\Omega}(K_t, X_t) [A^t - \frac{\sqrt{\Omega}}{2\sqrt{h}}],$$

$$\Psi_h^v = \frac{\partial \Psi(K_t, X_t, E_t V^{t+h}(\Gamma(K_t + hA^t(K_t, X_t) + \sqrt{h\Omega}(K_t, X_t) \varepsilon_{t+h}); h); h)}{\partial h}$$

Rewriting the above equation is:

$$0 = \max_{X_t \in \mathcal{X}^t(K_t)} \lim_{h \to 0} \left( \Psi_h^v + .5\Psi_t^v \left[ A^t [V^{t+h}_K (+) + V^{t+h}_K (-)] + [V^{t+h}_t (+) + V^{t+h}_t (-)] \right] + .5\Psi_t^v \sqrt{\Omega} [V^{t+h}_K (+) - V^{t+h}_K (-)] \right)$$
where
\[ V^{t+h}_K(\cdot) = V^{t+h}_K(K_t + hA^t + \sqrt{h}\Omega^t) , \]
\[ V^{t+h}_K(\cdot) = V^{t+h}_K(K_t + hA^t - \sqrt{h}\Omega^t) , \]
\[ V^{t+h}_t(\cdot) = V^{t+h}_t(K_t + hA^t + \sqrt{h}\Omega^t) , \]
\[ V^{t+h}_t(\cdot) = V^{t+h}_t(K_t + hA^t - \sqrt{h}\Omega^t) . \]

Thus,
\[ 0 = \max_{X_t \in \mathcal{X}(K_t)} \Psi^h_t(K_t, X_t, V^t; 0) + \Psi^t_t(K_t, X_t, V^t; 0) \{ A^tV^t_K(K_t) + V^t_t(K_t) \}
+ 0.5\Psi^t_t(K_t, X_t, V^t; 0)\sqrt{\Omega^t} \lim_{h \to 0} \left[ \frac{V^{t+h}_K(\cdot) - V^{t+h}_K(\cdot)}{2h} \right] \]

The fact that for all \( v_t, \Psi^t_t(K_t, X_t, v^t; 0) = v^t \) implies that
\[ \Psi^t_t(K_t, X_t, V^t; 0) = 1 . \]

By definition,
\[ \Psi^h_t(K_t, X_t, V^t; 0) = G(K_t, X_t, V^t) . \]

Thus, we have:
\[ 0 = \max_{X_t \in \mathcal{X}(K_t)} G(K_t, X_t, V^t) + \{ A^tV^t_K(K_t) + V^t_t(K_t) \} + 0.5\sqrt{\Omega^t} \lim_{h \to 0} \left[ \frac{V^{t+h}_K(\cdot) - V^{t+h}_K(\cdot)}{2h} \right] \]

Applying L’Hôpital’s rule yields:
\[ \lim_{h \to 0} \left[ \frac{V^{t+h}_K(\cdot) - V^{t+h}_K(\cdot)}{2h} \right] = \lim_{h \to 0} \left[ \frac{V^{t+h}_K(\cdot)A^t + \sqrt{\Omega^t}}{2h} - V^{t+h}_K(\cdot)A^t + \frac{\sqrt{\Omega^t}}{2h} \right] \]
\[ = V^{t}_K(K_t)\sqrt{\Omega^t}(K_t, X_t) \]

Plugging the result gives:
\[ 0 = \max_{X_t \in \mathcal{X}(K_t)} \left\{ G(K_t, X_t, V^t) + V^t_t(K_t) + V^t_K(K_t)A^t(K_t, X_t) + \frac{1}{2}V^t_K(K_t)\Omega^t(K_t, X_t) \right\} \]

Since \( X_t \) is not included in \( V^t_t(K_t) \), we can move it into the left-hand side to get the following continuous-time Bellman equation:
\[ -V^t_t(K_t) = \max_{X_t \in \mathcal{X}(K_t)} \left\{ G(K_t, X_t, V^t) + V^t_K(K_t)A^t(K_t, X_t) + \frac{1}{2}V^t_K(K_t)\Omega^t(K_t, X_t) \right\} . \]