Monotone Likelihood Ratio/Monotone Probability Ratio

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Monotone Likelihood Ratio

Given \( f_1(x) \) a probability density function, define the set of all probability density functions \( f_2(x) \) where the monotone likelihood ratio order is satisfied,

\[
A = \{ f_2(x) | f_2(x) = f_1(x) \varphi(x), \text{ where } \varphi(x) \text{ are positive increasing functions} \}.
\]

Then the convex hull of \( B \) as defined below is equals \( A \).

\[
B = \left\{ g \left| g(x; \xi) = \begin{cases} f_1(x) & \text{if } x \geq \xi \\ 0 & \text{otherwise} \end{cases} \right. \right\}.
\]

Proof:

Simply note that by definition, for every \( f_2(x) \in A \) there exists a \( \varphi(x) \), positive increasing function, such that \( f_2(x) = f_1(x) \varphi(x) \). Therefore, a basis set for \( A \) can be generated from the simple basis for the set of all positive increasing functions. Let

\[
C = \left\{ h \left| h(x; \xi) = \begin{cases} 1 & \text{if } x \geq \xi \\ 0 & \text{otherwise} \end{cases} \right. \right\}.
\]

The convex cone of \( C \) is the set of all positive increasing functions. One should note here that

\[
\Phi = \left\{ \varphi(x) \left| \varphi(x) = \frac{f_2(x)}{f_1(x)}, \forall f_2(x) \in A \right. \right\}
\]

is a proper subset of positive increasing functions, but we need not worry about this right now. Define

\[
\tilde{B} = \{ \tilde{g} | \tilde{g}(x; \xi) = f_1(x) h(x; \xi), \forall h(x; \xi) \in C \}.
\]

Then the convex cone of \( \tilde{B} \) contains \( A \). Now it remains to find the proper section of the convex cone of \( \tilde{B} \) which is \( A \). Note that \( f_2(x) \) is a probability density function, thus define the following set,

\[
B = \left\{ g \left| g(x; \xi) = \frac{f_1(x)}{1 - F_1(\xi)} h(x; \xi), \forall h(x; \xi) \in C \right. \right\}.
\]

As defined, all \( g(x; \xi) \in B \) are probability density functions. Furthermore, note that \( g(x; \xi) \) are extremal elements on the convex cone of \( \tilde{B} \); thus, the convex hull of \( B \) equals \( A \). QED.

The proof for Monotone Probability Ratio is structurally identical to above.

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