Let
\[ V_t(k_t) = \max_{x \in X_t(k_t)} u_t(k_t, x) + \beta E_t V_{t+1}(\tilde{k}_{t+1}) \]
\[ \tilde{k}_{t+1} = k_t + A(k_t, x, \tilde{z}_{t+1}) \]
(~ means random variable) for \( t = 0, 1, \ldots, T-1 \).

For \( T \),
\[ V_T(k_T) = \max_{x \in X_T(k_T)} u_T(k_T, x) \]

a) Show that if \( u_T(k_T, x) \) is monotonically increasing in \( k_T \) and \( X_T(\tilde{k}_T) \subseteq X_T(k_T) \) for all \( \tilde{k}_T > k_T \), then \( V_T(k_T) \) is monotonically increasing in \( k_T \).

Solution: Let \( \bar{x}_T = \arg \max_{x \in X_T(k_T)} u_T(\tilde{k}_T, x) \). Then:
\[ V_T(\tilde{k}_T) = \max_{x \in X_T(k_T)} u_T(\tilde{k}_T, x) \]
\[ \geq u_T(\tilde{k}_T, \bar{x}) \]
\[ \geq u_T(\tilde{k}_T, \bar{x}) \]
\[ = V_T(\tilde{k}_T). \]

because \( \bar{x} \subseteq X_T(\tilde{k}_T) \subseteq X_T(k_T) \). Therefore \( V_T(k_T) \) is monotonically increasing in \( k_T \).

b) Show that if \( V_{t+1}(k_{t+1}) \) is monotonically increasing in \( k_{t+1} \) and \( k_t + A(k_t, x, \tilde{z}_{t+1}) \) is monotonically increasing in \( k_t \) then \( u_t(k_t, x) + \beta E_t V_{t+1}(\tilde{k}_{t+1}) \) is monotonically increasing in \( k_t \) given \( x \).
Solution: \( u_t(k_t, x) + \beta E_t V_{t+1}(\tilde{k}_{t+1}) = u_t(k_t, x) + \beta E_t V_{t+1}(k_t + A(k_t, x, \tilde{z}_{t+1})) \).

If \( \hat{k}_T > \tilde{k}_T \), then by assumption \( \hat{k}_t + A(\hat{k}_t, x, z_{t+1}) \geq \tilde{k}_t + A(\tilde{k}_t, x, z_{t+1}) \), and therefore \( V_{t+1}(\hat{k}_t + A(\hat{k}_t, x, z_{t+1})) \geq V_{t+1}(\tilde{k}_t + A(\tilde{k}_t, x, z_{t+1}) \) for each \( x \) and \( z_{t+1} \) because \( V_{t+1}(\hat{k}_{t+1}) \) is monotonically increasing. So:
\[
\begin{align*}
& u_t(\hat{k}_t, x) + \beta E_t V_{t+1}(\hat{k}_t + A(\hat{k}_t, x, \tilde{z}_{t+1})) \\
& \geq u_t(\tilde{k}_t, x) + \beta E_t V_{t+1}(\tilde{k}_t + A(\tilde{k}_t, x, \tilde{z}_{t+1})) \\
& \geq u_t(\tilde{k}_t, \overline{x}_t) + \beta E_t V_{t+1}(\overline{k}_{t+1}(\tilde{k}_t, \overline{x}_t)) \\
& = V_t(\tilde{k}_t).
\end{align*}
\]

Therefore, \( V_t(\tilde{k}_t) \) is monotonically increasing in \( k_t \).

c) Suppose that \( V_{t+1}(k_{t+1}) \) is monotonically increasing in \( k_{t+1} \) and that \( u_t(k_t, x) \) and \( k_t + A(k_t, x, z_{t+1}) \) are monotonically increasing in \( k_t \) and finally that \( \chi_t(\hat{k}_t) \subset \chi_t(\tilde{k}_t) \) whenever \( \hat{k}_t > \tilde{k}_t \). Show that \( V_t(k_t) \) is monotonically increasing in \( k_t \).

Solution: By 1.b, \( u_t(k_t, x) + \beta E_t V_{t+1}(\tilde{k}_{t+1}(k_t, x)) \) is increasing in \( k_t \). Now, if \( \hat{k}_t > \tilde{k}_t \) and \( \overline{x}_t \in \arg \max_x u_t(\tilde{k}_t, x) + \beta E_t V_{t+1}(\tilde{k}_{t+1}(k_t, x)) \), then
\[
V_t(\hat{k}_t) = \max_x u_t(\hat{k}_t, x) + \beta E_t V_{t+1}(\hat{k}_{t+1}(\hat{k}_t, x)) \\
\geq u_t(\hat{k}_t, \overline{x}_t) + \beta E_t V_{t+1}(\hat{k}_{t+1}(\hat{k}_t, \overline{x}_t)) \quad \text{[because \( \overline{x}_t \in \chi_t(\hat{k}_t) \subset \chi_t(\tilde{k}_t) \)]} \\
\geq u_t(\tilde{k}_t, \overline{x}_t) + \beta E_t V_{t+1}(\tilde{k}_{t+1}(\tilde{k}_t, \overline{x}_t)) \\
= V_t(\tilde{k}_t).
\]

Therefore \( V_t(k_t) \) is monotonically increasing in \( k_t \).

d) Show that if \( u_t(k_t, x) \) and \( k_t + A(k_t, x, z_{t+1}) \) are monotonically increasing in \( k_t \) for all \( t = 0, 1, \ldots, T \) and \( \chi_t(\hat{k}_t) \subset \chi_t(\tilde{k}_t) \) whenever \( \hat{k}_t > \tilde{k}_t \) for all \( t = 0, 1, \ldots, T \), then \( V_t(k_t) \) is monotonically increasing in \( k_t \) for all \( t = 0, 1, \ldots, T \).

Solution: Part 1.b and 1.c together show that if \( V_{t+1}(k_{t+1}) \) is monotonically increasing in \( k_{t+1} \), then \( V_t(k_t) \) is monotonically increasing in \( k_t \). Part 1.a says that \( V_T(k_T) \) is monotonically increasing in \( k_T \). Therefore, by recursion, \( V_{T-1}(k_{T-1}), V_{T-2}(k_{T-2}) \ldots \)
\ldots \( V_0(k_0) \) are monotonically increasing in their arguments.

e) Modify this problem so that you can take a limit as the time interval gets short into a continuous time version of the problem. In the continuous time version of the problem, which assumption above is no longer necessary to assume separately since it is virtually guaranteed to be true as the time interval gets very short?
Solution: The new setup is:

\[ V_t(k_i) = \max_{x \in \mathcal{X}_t(k_i)} h u_t(k_t, x) + e^{-\rho h} E_t V_{t+h}(\tilde{k}_{t+h}) \]

\[ \tilde{k}_{t+h} = k_t + hA(k_t, x, \tilde{z}_{t+h}) \]

As \( h \to 0 \), the assumption that \( k_t + hA(k_t, x, \tilde{z}_{t+h}) \) is increasing in \( k_t \) for given \( x \) and \( \tilde{z}_{t+h} \) becomes easier and easier to satisfy (because the second term \( \to 0 \) as \( h \to 0 \)).

f) Consider a continuous time phase diagram with \( k \) on the horizontal axis and \( \lambda \) (the marginal value of \( k \)) on the vertical axis. Based on part 1.e, state a result about this phase diagram. Be clear about the assumptions needed to guarantee this result. Argue that this result is valid even though the problem is stochastic, so that the solution would be jumping around on the phase diagram.

Solution: \( V(k_i) \) monotonically increasing in \( k_i \) corresponds to \( \lambda = V'(k_i) \geq 0 \). So, we will be in the \( \lambda \geq 0 \) region of the phase diagram if \( u \) is monotonically increasing in \( k \) and \( \mathcal{X}_t(k_i) \subset \mathcal{X}_t(\hat{k}_i) \) whenever \( \hat{k}_i > k_i \). By part 1.e, the assumption \( k_i + hA(k_t, x, \tilde{z}_{t+h}) \) is monotonically increasing in \( k_i \) is guaranteed anyway as \( h \to 0 \).