Exercise 46
Use Proposition 16 to prove that an increase in wealth increases the demand for the risky asset if and only if the investor is DARA.

Solution 46
We want to determine the conditions under which the following holds:

\[ \forall \tilde{x}, E\tilde{x}u'(w_L + \tilde{x}) = 0 \Rightarrow \forall w_H > w_L : E\tilde{x}u'(w_H + \tilde{x}) \geq 0 \]

This is a straightforward restatement of Gollier equation (7.1),

\[ \forall \tilde{x} \in [a, b], Eg(\tilde{x})h(\tilde{x}, w_L) = 0 \Rightarrow \forall w_H > w_L : Eg(\tilde{x})h(\tilde{x}, w_H) \geq 0 \]

replacing \( g(\tilde{x}) = \tilde{x} \) and \( h(\tilde{x}, w) = u'(w + \tilde{x}) \).

Proposition 16 states that for any real valued function \( g \) that satisfies the single-crossing condition:

\[ \exists x_0 : \forall x : (x - x_0)g(x) \geq 0, \text{ the above equation holds if and only if } h \text{ is log-supermodular (LSPM).} \]

By Proposition 16, the above equation holds if and only if \( h(x, w) = u'(w + x) \) is LSPM. Furthermore, from Lemma 2, \( h(x, w) \) is LSPM if and only if \( [\partial h(x, w)/\partial x]/h(x, w) \) is non-decreasing in \( w \). That is,

\[ \frac{\partial}{\partial w} \left( \frac{\partial h(x, w)/\partial x}{h(x, w)} \right) \geq 0 \]

Replacing \( h(x, w) \) with \( u'(w + x) \) yields

\[ \frac{\partial}{\partial w} \left( \frac{u''(w + x)}{u'(w + x)} \right) \geq 0 \]

\[ \frac{\partial}{\partial w} \left( - \frac{u''(w + x)}{u'(w + x)} \right) \leq 0 \]

\[ \frac{\partial}{\partial w} A(w) \leq 0 \]

Which is the definition of Decreasing Absolute Risk Aversion (DARA).

We can see that the reverse is true as well because all of our results are "if and only if" results. \( u'(w + x) \) exhibits DARA if and only if \( u'(w + x) \) is LSPM. \( u'(w + x) \) is LSPM if and only if the above statement of Gollier equation (7.1) is true, which is the condition we intended to satisfy. This completes both directions of the proof.