Exercise 18  
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Consider an economy where all agents face an idiosyncratic risk to lose 100 with probability $p$. $N$ agents decide to create a mutual agreement where the aggregate loss in the pool will be equally split among its members.

- Describe the residual loss of each member of the pool when $N=2$ and $N=3$.
- Show that an increase from $N=2$ to $N=3$ generates a reduction in risk borne by each member of the pool, in the sense of SSD.

Let $a$ be the number of agents in the pool who lose 100, and let $x$ be the loss per agent in the pool (the residual loss). Then $x = \frac{100a}{N}$ and

$$
\Pr(x = \frac{100a}{N}) = f\left(\frac{100a}{N}\right) = \left(\frac{N}{a}\right) p^a (1-p)^{N-a} \quad \text{for } a = 0, 1, 2, ..., N
$$

For $N=2$,

Let $x_2$ be the residual loss, $f_2$ be the pdf of $x_2$, and $F_2$ be the cdf of $x_2$.

$f_2(0) = (1 - p)^2$

$f_2(50) = 2p(1-p)$

$f_2(100) = p^2$

$$
F_2(x) = \begin{cases} 
(1 - p)^2 & 0 \leq x < 50 \\
1 - p^2 & 50 \leq x < 100 \\
1 & x \geq 100
\end{cases}
$$

and $E\text{x}_2 = \frac{100}{2} \cdot \text{E}[\text{number of agents in pool who lose 100}] = \frac{100}{2} \cdot 2 \cdot p = 100p$

For $N=3$,

Let $x_3$ be the residual loss, $f_3$ be the pdf of $x_3$, and $F_3$ be the cdf of $x_3$.

$f_3(0) = (1 - p)^3$

$f_3\left(\frac{100}{3}\right) = 3p(1-p)^2$

$f_3\left(\frac{200}{3}\right) = 3p^2(1-p)$

$f_3(100) = p^3$
To demonstrate that an increase from \( N=2 \) to \( N=3 \) generates a reduction in risk borne by each member in the sense of SSD, we will show that the Rothschild-Stiglitz condition holds:

\[
\int_{0}^{x} F_3(\theta) d\theta \leq \int_{0}^{x} F_2(\theta) d\theta \quad \forall x \in [0,100]
\]

Or, equivalently:

\[
\int_{0}^{x} F_2(\theta) d\theta - \int_{0}^{x} F_2(\theta) d\theta \geq 0 \quad \forall x \in [0,100]
\]

The graph below suggests that this difference will reach a minimum at either \( x=50 \) or \( x=100 \). Therefore, if the Rothschild-Stiglitz condition holds for \( x=50 \) and \( x=100 \), then it will hold for \( \forall x \in [0,100] \).

\[
\int_{0}^{50} F_3(\theta) d\theta = \frac{100}{3} (1 - p)^3 + (1 - p)^2 (1 + 2p)(50 - \frac{100}{3})
\]

\[
= \frac{50}{3} (1 - p)^2 [2(1 - p) + 1 + 2p]
\]

\[
= 50(1 - p)^2 = \int_{0}^{50} F_2(\theta) d\theta
\]

And

\[
\int_{50}^{100} F_3(\theta) d\theta = (\frac{200}{3} - 50)(1 - p)^2 (1 + 2p) + (100 - \frac{200}{3})(1 - p^3)
\]

\[
= \frac{50}{3} [3 - 3p^2]
\]

\[
= 50(1 - p^2) = \int_{50}^{100} F_2(\theta) d\theta
\]
Therefore,  
\[ \int_{0}^{100} F_3(\theta) \, d\theta = \int_{0}^{50} F_3(\theta) \, d\theta + \int_{50}^{100} F_3(\theta) \, d\theta = \int_{0}^{50} F_2(\theta) \, d\theta + \int_{50}^{100} F_2(\theta) \, d\theta = \int_{0}^{100} F_2(\theta) \, d\theta \]

We conclude that  
\[ \int_{0}^{x} F_3(\theta) \, d\theta \leq \int_{0}^{x} F_2(\theta) \, d\theta \quad \forall x \in [0,100] \]

and that an increase from \( N=2 \) to \( N=3 \) generates a reduction in risk borne by each member of the pool in the sense of SSD.