(1) Suppose that instead of having a free choice of alpha, that the share of risky assets \( s = \alpha / w \) is given exogenously (you don’t have any choice about it). Solve the Merton model for stationary \( s, \mu, \sigma, r \) and the terminal time \( T \). (Note: This is a slight modification of original problem, which allowed the exogenous parameters to be time-varying.)

After fixing \( \alpha = ws \) we solve a restricted Merton model

\[
\tilde{V}(w_0, 0) = \max_c E_0 \int_0^T e^{-\rho t} \frac{c^{1-\gamma}}{1-\gamma} dt
\]

subject to

\[
dw = [rw + ws \mu - c]dt + ws \sigma dz; \ w_0 \ \text{given}
\]

where \( \tilde{V}(w, t) \) is the value function starting at time \( t \) with wealth \( w \). The tilde on \( \tilde{V} \) denotes that the value function is restricted by fixed \( s \). We can write the Bellman equation of this problem as

\[
\rho \tilde{V}(w, t) - \tilde{V}_t(w, t) = \max_c \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \tilde{V}_w(w, t)[rw + ws \mu - c] + \frac{1}{2} \tilde{V}_{ww}(w, t)w^2 s^2 \sigma^2 \right\}.
\]

Note that this problem admits the following scale symmetry:

\[
w \longrightarrow \theta w \\
c \longrightarrow \theta c \\
\tilde{V} \longrightarrow \theta^{1-\gamma} \tilde{V}
\]

where \( \tilde{V} \) is the maximized (restricted) value function. Applying the symmetry theorem we have

\[
\theta^{1-\gamma} \tilde{V}(w, t) = \tilde{V}(\theta w, t),
\]

which implies that

\[
\tilde{V}(w, t) = w^{1-\gamma} \tilde{V}(1, t)
\]

for \( \theta = 1/w \). Differentiation then yields the following identities:
\[\bar{V}_t(w, t) = w^{1-\gamma}\bar{V}_t(1, t),\]
\[\bar{V}_w(w, t) = (1 - \gamma)w^{-\gamma}\bar{V}(1, t), \text{ and} \]
\[\bar{V}_{ww}(w, t) = -\gamma(1 - \gamma)w^{-\gamma-1}\bar{V}(1, t).\]

Substituting back into the Bellman equation above we have
\[
\rho w^{1-\gamma}\bar{V}(1, t) - w^{1-\gamma}\bar{V}_t(1, t) = \max_c \left\{ \frac{c^{1-\gamma}}{1-\gamma} + (1 - \gamma)w^{-\gamma}\bar{V}(1, t)[rw + ws\mu - c] \right. \\
\left. - \frac{1}{2}\gamma(1 - \gamma)w^{-\gamma-1}\bar{V}(1, t)w^2s^2\sigma^2 \right\}.
\]

The first-order condition with respect to \(c\) is

\[c^{-\gamma} = (1 - \gamma)w^{-\gamma}\bar{V}(1, t),\]

which after simplification yields

\[c = A(t)w,\]

where \(A(t) = [(1 - \gamma)\bar{V}(1, t)]^{-1/\gamma}\) is the average propensity to consume at time \(t\). Note that

\[\bar{V}(1, t) = \frac{A(t)^{-\gamma}}{1 - \gamma},\]

so

\[\bar{V}(w, t) = w^{1-\gamma}\frac{A(t)^{-\gamma}}{1 - \gamma}.\]

Differentiation then yields the following identities:

\[\bar{V}_t(w, t) = -\gamma w^{1-\gamma}\frac{A(t)^{-\gamma-1}}{1 - \gamma}A(t),\]
\[\bar{V}_w(w, t) = w^{-\gamma}A(t)^{-\gamma}, \text{ and} \]
\[\bar{V}_{ww}(w, t) = -\gamma w^{-\gamma-1}A(t)^{-\gamma}.\]

Substituting into the Bellman equation above we have

\[
\rho w^{1-\gamma}\frac{A(t)^{-\gamma}}{1 - \gamma} + \gamma w^{1-\gamma}\frac{A(t)^{-\gamma-1}}{1 - \gamma}A(t) = \left[\frac{A(t)w^{1-\gamma}}{1 - \gamma}\right] + w^{-\gamma}A(t)^{-\gamma}[rw + ws\mu - A(t)w] \\
- \frac{1}{2}\gamma w^{-\gamma-1}A(t)^{-\gamma}w^2s^2\sigma^2.
\]
Dividing by \( w^{1-\gamma}A(t)^{-\gamma}/(1 - \gamma) \) on both sides yields

\[
\rho + \gamma \frac{\dot{A}(t)}{A(t)} = A(t) + (1 - \gamma)[r + s\mu - A(t)] - \frac{\gamma(1 - \gamma)}{2}s^2\sigma^2
\]

\[
= \gamma A(t) + (1 - \gamma)\left[r + s\mu - \frac{\gamma}{2}s^2\sigma^2\right],
\]

which after dividing by \( \gamma \) on both sides gives us

\[
\frac{\rho}{\gamma} + \frac{\dot{A}(t)}{A(t)} = A(t) + \left(\frac{1}{\gamma} - 1\right)\left[r + s\mu - \frac{\gamma}{2}s^2\sigma^2\right].
\]

Now note that because the exogenous parameters are time-invariant the original problem admits the following time-shift symmetry:

\[
\begin{align*}
\lambda &
\rightarrow t + \lambda \\
T &
\rightarrow T + \lambda \\
\mu(t) &
\rightarrow \mu(t + \lambda) \\
\sigma(t) &
\rightarrow \sigma(t + \lambda) \\
r(t) &
\rightarrow r(t + \lambda) \\
w(t) &
\rightarrow w(t + \lambda) \\
c(t) &
\rightarrow c(t + \lambda) \\
\tilde{V}(t) &
\rightarrow \tilde{V}(t + \lambda).
\end{align*}
\]

Combining the time-shift symmetry with the scale symmetry from above we have

\[
A(t) = A(t + \lambda)
\]

for all \( \lambda \) since the agent’s same optimal choices are just shifted in time by \( \lambda \). If we let \( \lambda = -t \) then

\[
A(t) = A(0),
\]

which implies that \( \dot{A}(t) = 0 \) so

\[
A = \frac{\rho}{\gamma} + \left(1 - \frac{1}{\gamma}\right)\left[r + s\mu - \frac{\gamma}{2}s^2\sigma^2\right].
\]

(2) Discuss how the solution to (1) can help you understand the solution to the Merton problem in cases of (a) no constraints at all, (b) short-sale constraints, (c) borrowing constraints, and (d) a limitation to only two choices for the share \( s \).

(a) We know from (1) above that
\[ \tilde{V}(w, t) = w^{1-\gamma} A(t)^{-\gamma}. \]

Substituting for \( A \) from our solution to (1) we have

\[ \tilde{V}(w, t) = \frac{w^{1-\gamma}}{1-\gamma} \left( \frac{\rho}{\gamma} + \left( 1 - \frac{1}{\gamma} \right) \left[ r + s \mu - \frac{\gamma}{2} s^2 \sigma^2 \right] \right)^{-\gamma}. \]

Now suppose that the agent is given free choice over \( s \). The solution to the Merton problem with no constraints is just

\[ V(w, t) = \max_s \tilde{V}(w, t). \]

The optimal \( s \) can be found by solving the following optimization sub-problem:

\[ \max_s \left[ r + s \mu - \frac{\gamma}{2} s^2 \sigma^2 \right] \]

since \( \tilde{V}(w, t) \) is strictly increasing in \( r + s \mu - \frac{\gamma}{2} s^2 \sigma^2 \) for all \( \gamma \). It can be shown that the resulting two-step solution (i.e., first solving for \( c \) given fixed \( s \) and then optimizing over \( s \)) is identical to the solution to the standard Merton problem.

(b) The same as (a) but add the constraint that \( \alpha \geq 0 \implies ws \geq 0 \).

(c) The same as (a) but add the constraint that \( \alpha + c \leq f(w) + w \implies ws + c \leq f(w) + w \), where \( f(w) \) is the agent’s available credit line conditional on his wealth \( w \).

(d) The same as (a) but now maximize over the constraint set \( s \in \{ s_L, s_H \} \) instead of all \( s \).