Exercise 6: Consider the utility function

\[ u(z) = \begin{cases} 
  z & \text{for } z \leq z_0 \\
  z_0 + \alpha (z - z_0) & \text{for } z > z_0 
\end{cases} \]

with a positive constant \( \alpha < 1 \).

1. Show that \( u(\cdot) \) is concave.

2. Show that \( u(\cdot) \) exhibits first-order risk aversion at \( z = z_0 \), namely that \( \pi(z_0, u(\cdot), k\bar{x}) \) tends to zero when \( k \) tends to zero as \( k \), rather than as \( k^2 \).

3. Show that a reduction in the constant \( \alpha \) increases the degree of risk aversion, in the sense of Arrow-Pratt.

4. Does \( u(\cdot) \) satisfy DARA?
Part 1: Since $u(\cdot)$ is not differentiable at $z = z_0$, we cannot use calculus to show the function is everywhere concave. So, we go back to the definition of a concave function. A function $f$ is concave if and only if for all $\lambda \in [0, 1]$ and all pairs $(x, y)$ in the domain of $f$,

$$\lambda f(x) + (1 - \lambda) f(y) \leq f(\lambda x + (1 - \lambda) y)$$

There are three cases to consider:

- **Case 1**: $x, y \leq z_0$ - We can see, because the utility function is linear over this range,

$$\lambda u(x) + (1 - \lambda) u(y) = \lambda x + (1 - \lambda) y$$

The weak inequality $\lambda u(x) + (1 - \lambda) u(y) \leq u(\lambda x + (1 - \lambda) y)$ is satisfied with equality for all $x, y \leq z_0$.

- **Case 2**: $x, y \geq z_0$ - For the same reason as in Case 1, the weak inequality is satisfied with equality.

- **Case 3**: $x \leq z_0$ and $y \geq z_0$ - First, we see that

$$\lambda u(x) + (1 - \lambda) u(y) = \lambda [z_0 + \alpha (x - z_0)] + (1 - \lambda) [z_0 + \alpha (y - z_0)]$$

Next, we have two sub-cases to consider, one where the expected value of the lottery is (weakly) less than $z_0$ and one where it (weakly) exceeds $z_0$. We address each in turn.

- **Case 3.1**: For all $\lambda$ such that $\lambda x + (1 - \lambda) y \leq z_0$, we have

$$u(\lambda x + (1 - \lambda) y) = \lambda x + (1 - \lambda) y$$

So,

$$\lambda u(x) + (1 - \lambda) u(y) - u(\lambda x + (1 - \lambda) y) = (1 - \alpha)(1 - \lambda)(z_0 - y) \leq 0$$

since $y \geq z_0$ by assumption.

- **Case 3.2**: For all $\lambda$ such that $\lambda x + (1 - \lambda) y \geq z_0$, we have

$$u(\lambda x + (1 - \lambda) y) = (1 - \alpha)z_0 + \alpha(\lambda x + (1 - \lambda) y)$$

So,

$$\lambda u(x) + (1 - \lambda) u(y) - u(\lambda x + (1 - \lambda) y) = (1 - \alpha)\lambda(x - z_0) \leq 0$$

since $x \leq z_0$ by assumption. ■
**Part 2:** The risk premium $\pi$ is the amount of money an agent is willing to pay to make him indifferent between taking the gamble which yields $E[u(w_0 + k\tilde{x})]$ and receiving $u(w_0 - \pi)$ with certainty. The risk premium $\pi(z, u(\cdot), k\tilde{x}) = g(k)$ at $z = z_0$ is found by solving the following equation:

$$E[u(z_0 + k\tilde{x})] = u(z_0 - g(k))$$

First, we note that since $u(\cdot)$ is concave, the risk premium is non-negative; indeed, at $z_0$ the risk premium is strictly positive for a mean zero lottery (see below). Therefore, $u(z_0 - g(k)) = z_0 - g(k)$ since, for all $z \leq z_0$, $u(z) = z$.

This implies

$$g(k) = z_0 - E[u(z_0 + k\tilde{x})]$$

We need to determine the expected utility of the lottery $\tilde{x}$ to get the risk premium. Specifically, we consider a mean zero lottery, i.e., $E[\tilde{x}] = 0$. So,

$$g(k) = z_0 - \int u(z_0 + kx) dF(x)$$

$$= z_0 - \left\{ \int_{-\infty}^{0} (z_0 + kx) dF(x) + \int_{0}^{\infty} ((1 - \alpha)z_0 + \alpha(z_0 + kx)) dF(x) \right\}$$

$$= z_0 - \left\{ \int_{-\infty}^{0} z_0 dF(x) + k \int_{-\infty}^{0} x dF(x) + \int_{0}^{\infty} z_0 dF(x) + \alpha k \int_{0}^{\infty} x dF(x) \right\}$$

$$= z_0 - \left\{ \int_{-\infty}^{\infty} z_0 dF(x) + k \int_{-\infty}^{0} x dF(x) + \alpha k \int_{0}^{\infty} x dF(x) \right\}$$

$$= z_0 - z_0 - k \left\{ \int_{-\infty}^{0} x dF(x) + \alpha \int_{0}^{\infty} x dF(x) \right\}$$

$$= -k \left\{ \int_{-\infty}^{0} x dF(x) + \alpha \int_{0}^{\infty} x dF(x) \right\}$$

$$= -k \left\{ E[x] - \int_{0}^{\infty} x dF(x) + \alpha \int_{0}^{\infty} x dF(x) \right\}$$

$$= k(1 - \alpha) \int_{0}^{\infty} x dF(x)$$

Since we have $\tilde{x}$ is non-degenerate, $E[\tilde{x}] = 0$ implies $\int_{0}^{\infty} x dF(x)$ is positive and finite. Then, since $\alpha \in (0, 1)$, the risk premium at $z_0$ for a mean zero lottery is strictly positive, i.e., $g(k) > 0$.

Therefore, we can see as $k$ goes to zero the risk premium goes to zero, i.e.,

$$\lim_{k \to 0} g(k) = 0$$

We also see that the risk premium goes to zero as $k$

$$0 < \lim_{k \to 0^+} \frac{g(k)}{k} = (1 - \alpha) \int_{0}^{\infty} x dF(x) < \infty$$
Part 3: To show that a reduction in the constant $\alpha$ increases the degree of risk aversion, in the sense of Arrow-Pratt, we want to show that decreasing $\alpha$ is akin to transforming $u(z)$ with an increasing, concave function. This appeals to Proposition 2 on page 20 of Gollier.

We are starting off with the function

$$u_1(z) = \begin{cases} 
  z & \text{for } z \leq z_0 \\
  z_0 + \alpha (z - z_0) & \text{for } z > z_0
\end{cases}$$

and moving to the function

$$u_2(z) = \begin{cases} 
  z & \text{for } z \leq z_0 \\
  z_0 + \alpha' (z - z_0) & \text{for } z > z_0
\end{cases}$$

where $\alpha' < \alpha$.

Consider the function

$$\phi(u_1) = \begin{cases} 
  u_1 & \text{for } u_1(z) \leq u_1(z_0) \\
  (1 - \beta) z_0 + \beta u_1 & \text{for } u_1(z) > u_1(z_0)
\end{cases}$$

where $\beta \in (0, 1)$.

Next, we see

$$\phi(u_1(z)) = \begin{cases} 
  z & \text{for } z \leq z_0 \\
  (1 - \beta) z_0 + \beta (z_0 + \alpha (z - z_0)) = z_0 + \beta \alpha (z - z_0) & \text{for } z > z_0
\end{cases}$$

Since $\beta \in (0, 1), \beta \alpha < \alpha$. Letting $\alpha' = \alpha \beta$ yields the result that $u_2 = \phi(u_1)$. So, $\phi(\cdot)$ is an increasing transformation of $u_1$, and, from the first part of this exercise, we know that functions that take the form of $\phi(\cdot)$ are concave. So, a reduction in $\alpha$ increases the degree of risk aversion because it can result from transforming $u_1$ with an increasing, concave function.

$\blacksquare$
Part 4: On page 24, Gollier defines decreasing absolute risk aversion (DARA): Preferences exhibit decreasing absolute risk aversion if the risk premium associated to any risk is a decreasing function of wealth: \( \partial \pi(w_0, u(\cdot), \tilde{x})/\partial w_0 \leq 0 \) for any \( w_0, \tilde{x} \).

Since this condition is supposed to hold for any initial wealth \( w_0 \) and any lottery \( \tilde{x} \), to show \( u(\cdot) \) does not exhibit DARA, it suffices to show that for a particular lottery and initial wealth, the risk premium is increasing in initial wealth.

Consider the lottery that yields \(-x\) with probability 0.5 and yields \(+x\) with probability 0.5. Consider any \( x \) and \( w_0 \) where \( w_0 < z_0 \) and \( w_0 + x \leq z_0 \). Then the risk premium must be zero at these levels of initial wealth. Formally, the risk premium \( \pi \) is defined by

\[
E[u(w_0 + \tilde{x})] = u(w_0 - \pi)
\]

Since \( w_0 < z_0 \) by construction and \( \pi \geq 0 \) by the concavity of \( u(\cdot) \), \( u(w_0 - \pi) = w_0 - \pi \) since \( u(z) = z \) for all \( z \leq z_0 \).

Therefore,

\[
0.5(w_0 - x) + 0.5(w_0 + x) = w_0 = w_0 - \pi
\]

which implies \( \pi = 0 \).

Now, consider a new initial level of wealth \( w'_0 > w_0 \) such that \( w'_0 < z_0 \) and \( w'_0 + x > z_0 \). The risk premium \( \pi \) is defined by

\[
0.5(w'_0 - x) + 0.5((1 - \alpha)z_0 + \alpha(w'_0 + x)) = w'_0 - \pi
\]

After some rearrangement, we get

\[
\pi = 0.5(1 - \alpha)(w'_0 + x - z_0) > 0
\]

since \( w'_0 + x > z_0 \) by construction and \( \alpha \in (0, 1) \).

Thus, we have found a particular lottery that yields no risk premium at low levels of wealth \( (w_0) \) and a positive risk premium at certain greater levels of wealth \( (w'_0) \). This proves \( u(\cdot) \) does not exhibit decreasing absolute risk aversion. ■