Exercise 4: Risk aversion is a concept that is independent of the expected utility model. As an illustration of this, consider the rank-dependent EU model described by equation (1.4). Assume that \( u(z) = z \). Characterize the set of functions \( g(\cdot) \) that make the decision maker risk-averse (see Cohen 1995).

We are considering the lottery \( L = (x_1, p_1; x_2, p_2; \ldots; x_S, p_S) \), with \( x_1 < x_2 < \cdots < x_S \). Equation (1.4) in Gollier reads

\[
U(L) = \sum_{s=1}^{S} u(x_s) \left[ g \left( \sum_{t=1}^{s} p_t \right) - g \left( \sum_{t=1}^{s-1} p_t \right) \right] \tag{1}
\]

with \( g(0) = 0 \) and \( g(1) = 1 \).

In our case, we have

\[
U(L) = \sum_{s=1}^{S} x_s \left[ g \left( \sum_{t=1}^{s} p_t \right) - g \left( \sum_{t=1}^{s-1} p_t \right) \right] \tag{2}
\]

The expected value of the lottery is

\[
E[L] = \sum_{s=1}^{S} p_s x_s = \mu(L) \tag{3}
\]

which is also equal to the utility of the expected value of the lottery, i.e., \( E[L] = U(E[L]) \). For a risk averse agent, \( U(L) \leq U(E[L]) \). This implies

\[
U(L) - \sum_{s=1}^{S} p_s x_s \leq 0 \tag{4}
\]

The left-hand side is

\[
\sum_{s=1}^{S} x_s \left[ g \left( \sum_{t=1}^{s} p_t \right) - g \left( \sum_{t=1}^{s-1} p_t \right) - p_s \right] = \sum_{s=1}^{S} x_s \left[ (g \left( \sum_{t=1}^{s} p_t \right) - \sum_{t=1}^{s} p_t) - (g \left( \sum_{t=1}^{s-1} p_t \right) - \sum_{t=1}^{s-1} p_t) \right] \tag{5}
\]
Expanding this sum will be helpful:

\[ x_1 [(g(p_1) - p_1) - (g(0) - 0)] + x_2 [(g(p_1 + p_2) - (p_1 + p_2)) - (g(p_1) - (p_1))] + \ldots \\
+ x_S [(g(1) - 1) - (g(p_1 + \cdots + p_{S-1}) - (p_1 + \cdots + p_{S-1}))] \quad (6) \]

Rearranging, we have

\[(x_1 - x_2)(g(p_1) - p_1) + (x_2 - x_3)(g(p_1 + p_2) - (p_1 + p_2)) + \cdots + (x_S - x_S) (g(p_1 + \cdots + p_{S-1}) - (p_1 + \cdots + p_{S-1})) \quad (7)\]

This can be rewritten more compactly as

\[ \sum_{s=1}^{S-1} (x_s - x_{s+1}) \left( g \left( \sum_{t=1}^{s} p_t \right) - \sum_{t=1}^{s} p_t \right) \quad (8) \]

Now, since \( x_s < x_{s+1} \) for all \( s \), and the support of \( g(\cdot) \) is \([0, 1]\) with \( g(0) = 0 \) and \( g(1) = 1 \), a sufficient condition for this sum to be nonpositive is for \( g(z) \geq z \) for all \( z \in [0, 1] \) and \( g(0) = 0 \) and \( g(1) = 1 \).\(^1\)

It turns out this condition is also necessary. To see why, consider some function \( g(\cdot) \) that is below \( z \) on some part of the support and above \( z \) on some other part of the support. Recall, the condition had to hold for any arbitrary set of \( x' \)'s such that \( x_s < x_{s+1}, s = 1, \ldots, S - 1 \). Now consider some \( x_{s^*} \) and \( x_{s^*+1} \) such that \( (g(p_1 + \cdots + p_{s^*}) - (p_1 + \cdots + p_{s^*})) < 0 \). So

\( (x_{s^*} - x_{s^*+1}) (g(p_1 + \cdots + p_{s^*}) - (p_1 + \cdots + p_{s^*})) > 0 \). Since the only restriction on the \( x' \)'s is \( x_s < x_{s+1}, s = 1, \ldots, S - 1 \), if \( x_{s^*+1} - x_{s^*} \) is arbitrarily large, and the the other \( (x_{s+1} - x_s) \) are arbitrarily small, the whole summation becomes positive. So, \( g(z) \geq z \) for all \( z \in [0, 1] \) and \( g(0) = 0 \) and \( g(1) = 1 \) is necessary.

We can then write any lottery \( L \) as offering \( \mu(L) \) with certainty plus a mean zero lottery \( \tilde{L} \), where \( \tilde{L} = (x_1 - \mu(L), p_1; x_2 - \mu(L), p_2; \ldots; x_S - \mu(L), p_S) \). We then have

\[ U(L) = U(\mu(L) + \tilde{L}) \leq U(\mu(L)) \quad (9) \]

Since \( \tilde{L} \) is a mean zero risk and this inequality holds for all \( \mu(L) \), we have that the agent dislikes all zero-mean risks at all wealth levels, which is Gollier’s definition of a risk averse agent.

\(^1\)Notice, this includes all increasing and concave functions on this support that satisfy the boundary conditions.