Recap, and outline of Lecture 8

Previously

- Posed several questions re. existence of extreme points of polyhedra and existence of optimal solutions of LPS:
  1. When do polyhedra have (and don’t have) extreme points?
  2. Do (feasible, bounded) LPS always have optimal solutions?
  3. Can an optimal solution always be found at an extreme point of the feasible region?
    - …and what if the feasible region doesn’t have any extreme points?
- Answered question 1 (Theorem 2.6): Non-empty polyhedron has extreme points if and only if it does not contain a line.
  - Algebraically: if and only if $n$ of $a_i$'s are lin. ind. for
    $P = \{x \in \mathbb{R}^n \mid a'_i x \geq b_i, \ i = 1,\ldots, m\}$
- Partially answered question 3 (Theorem 2.7): If $P$ has extreme points and $\min\{c'x \mid x \in P\}$ has an optimal solution, then it has an opt. sol. which is an extreme point of $P$.

Today

- What happens if we relax assumptions of Theorem 2.7?
- LPS in standard form

Optimality of extreme points

**Theorem 2.8**

Consider the linear programming problem of minimizing $c'x$ over a polyhedron $P$. Suppose that $P$ has at least one extreme point. Then, either the optimal cost is equal to $-\infty$, or there exists an extreme point which is optimal.

- Unlike in Theorem 2.7, we do not assume that the problem has an optimal solution
- In a general (non-linear) optimization problem, having a bounded optimal cost does not imply that an optimal solution exists
  - Recall $\min \frac{1}{x}$ s.t. $x \geq 0$
- Assume the optimal cost is finite. Follow the same steps as Theorem 2.6, (b) $\Rightarrow$ (a), except when picking $d$, ensure $c'd \leq 0$.
  - Given $x \in P$, we can find a BFS with cost $\leq c'x$. 
Existence of optimal solutions in general LPs

- Suppose, we are given a general LP problem (in minimization form). What possible solution status can it have?
  - The LP might be infeasible (feasible region empty)
  - If the LP is feasible, it might be unbounded (i.e., the objective function is unbounded below)
  - If the LP is feasible and not unbounded, and if the feasible region has extreme points, it has one or more optimal solutions (at least one of which can be found at an extreme point of the feasible region)
  - What if the feasible region doesn’t have extreme points?
    - ... or if we are unable to determine whether it has extreme points?
- We would like to find out whether an arbitrary LP problem that is not infeasible or unbounded has an optimal solution.
- Approach: given an (arbitrary) LP problem, convert it into an equivalent LP in standard form

“Standard form” LP problems

**Standard form LPs**

A linear programming problem of the form

\[
\begin{align*}
\text{min} & \quad c'x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

is said to be in **standard form**.

- All variables \(x_j\) satisfy \(x_j \geq 0, \ j = 1, \ldots, n\);
- All (other) constraints are linear equalities
  \(a_i'x = b_i, \ i = 1, \ldots, m\)
- Sometimes, refer to the polyhedron
  \(P = \{x \in \mathbb{R}^n \mid Ax = b, \ x \geq 0\}\) as being in standard form
- **Corollary 2.2, part II** Every nonempty polyhedron in standard form has at least one basic feasible solution.
Conversion into standard form

Any LP can be re-written as an equivalent LP in standard form

- **Elimination of free variables:**
  - If variable $x_j$ is unrestricted in sign in the original LP,
    - define two new variables: $x_j^+ \geq 0$ and $x_j^- \geq 0$
    - replace every occurrence of $x_j$ (in constraints and objective) with $x_j^+ - x_j^-$
  - Variable $x_j \leq 0$?

- **Elimination of inequality constraints:**
  - If the original LP has a constraint in the form $a_i^T x \leq b_i$,
    - define a new variable $s_i \geq 0$
    - re-write the constraint as $a_i^T x + s_i = b_i$
    - $s_i$ does not appear in the objective function (i.e., has coefficient of 0)
    - $s_i$ is called a *slack* variable
  - Constraint in the form $a_i^T x \geq b_i$?

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**Claim:** the original LP (OLP) and the LP in standard form obtained through above conversion steps (SLP) are equivalent.

**Proof outline:**

- Assume, wolog, that (OLP) has the form

  \[
  \text{(OLP)} \quad \min \quad c^T x \\
  \text{s.t.} \quad a_i^T x \leq b_i, \ i = 1, \ldots, m
  \]

  ("wolog" = "without loss of generality")

- Then (SLP) has the form

  \[
  \text{(SLP)} \quad \min \quad c^T x^+ - c^T x^- \\
  \text{s.t.} \quad a_i^T x^+ - a_i^T x^- + s_i = b_i, \ i = 1, \ldots, m \\
  x^+, x^-, s \geq 0
  \]

- Let $(x^+, x^-, s)$ be a feasible solution of (SLP). (OLP) has a feasible solution with the same obj. value. Thus, $\text{value}_{\text{OLP}} \leq \text{value}_{\text{SLP}}$

- Let $x$ be a feasible solution of (OLP). (SLP) has a feasible solution with the same obj. value. Thus, $\text{value}_{\text{SLP}} \leq \text{value}_{\text{OLP}}$
Existence of solutions to LPs

**Corollary 2.3**

Consider the linear programming problem of minimizing $c^T x$ over a nonempty polyhedron. Then, either the optimal cost is equal to $-\infty$, or there exists an optimal solution.

- Note: no assumptions on whether or not feasible region has any extreme points
- To see why this is true:
  - Convert the LP into an equivalent LP in standard form
  - The standard from LP's feasible region has at least one extreme point
  - By Theorem 2.8,
    - either the standard form LP is unbounded (and then so is the original),
    - or it has an (extreme point) optimal solution, which can be converted to an optimal solution of the original LP
- Standard form LPs also more convenient computationally

Caterer problem


- A catering company has upscale events of various sizes planned for each of the following several weekends. They need a cost-efficient plan to manage their needs for fabric napkins for this period.
  - For weekday events, they use paper napkins
- The size of each event (and hence the number of required napkins) is known
- For each weekend, they can buy new napkins
- Used (i.e., dirty) napkins can be sent to be laundered, but this process takes several weeks
- Used napkins can be sent to an express laundry, which returns them faster than regular one (but charges more)
Data/information needed for the catering problem

- $c_P$ is the cost of purchasing a new napkin
- $c_L$ is the cost of laundering a used napkin, and $p > 1$ is the number of weeks it takes
- $c_E$ is the cost of express-laundering a used napkins, and $q < p$ is the number of weeks it takes
- $n$ is the number of weeks the company is planning for
- $r_j$, $j = 1, \ldots, n$ is the number of napkins required in week $j$

Additional information needed:

- How many clean/dirty napkins do we have prior to week 1?
- What happens to clean/dirty napkins after week $n$?

My assumptions:

- $n$ weeks represents a “party season.” Each season, the menu and the decorating scheme that goes with it are changed.
- Therefore, the company has to buy all new napkins prior to week 1, and gets rid of napkins on hand after week $n$ (at negligible cost).

CP Variables

Each week, the company needs to decide

- Prior to the event:
  - How many new napkins to purchase: $x_j$, $j = 1, \ldots, n$
- Immediately after the event:
  - How many used napkins to launder: $y_j$, $j = 1, \ldots, n$
  - How many used napkin to express-launder: $z_j$, $j = 1, \ldots, n$

For convenience, let’s define additional variables:

- $s_j$, $j = 1, \ldots, n$ — dirty napkins carried over from week $j$ to week $j + 1$
  - In a typical week $j$,
    \[ s_{j-1} + r_j = y_j + z_j + s_j \]

- $t_j$, $j = 1, \ldots, n$ — clean napkins carried over from week $j$ to week $j + 1$
  - In a typical week $j$,
    \[ x_j + t_{j-1} + y_{j-p} + z_{j-q} = r_j + t_j \]
CP Formulation

Take into account:

- Need to have at least $r_j$ clean napkins on hand in week $j$
- No carryover (clean or dirty) into week 1
- Nothing coming from express laundry in the first $q$ weeks
- Nothing coming from the regular laundry in the first $p$ weeks

$$\min_{x,y,z,s,t} \sum_{j=1}^{n} (c^P x_j + c^L y_j + c^E z_j)$$

s.t. $r_1 = y_1 + z_1 + s_1$

$s_{j-1} + t_j = y_j + z_j + s_j, \quad j = 2, \ldots, n$

$x_1 = r_1 + t_1$

$x_j + t_{j-1} = r_j + t_j, \quad j = 2, \ldots, q$

$x_j + t_{j-1} + z_{j-q} = r_j + t_j, \quad j = q + 1, \ldots, p$

$x_j + t_{j-1} + y_{j-p} + z_{j-q} = r_j + t_j, \quad j = p + 1, \ldots, n$

$x_j, y_j, z_j, s_j, t_j \geq 0, \text{ (integer?) } \quad j = 1, \ldots, n$

- “Inventory balance” constraint groups (2 dirty, $\geq 4$ clean)
- How does this formulation ensure satisfaction of demand?

CP Extensions

- See files `caterer.mod` and `caterer.dat`
- With easy modifications, above formulation can accommodate
  - Availability of dirty, clean and “in process” laundered napkins before week 1
  - Capacity restrictions on laundry services
  - Different purchase/service costs in each week
  - Weekly charges for storage of clean and dirty napkins
  - Cost (revenue) for disposal (resale) of remaining napkins at the end of the season
- We’ll see later in the course why integrality assumptions are not required for this problem