Recap, and outline on Lecture 16

Previously

- Defined a dual of a (minimization) LP
- Motivated via
  - Economic/market considerations
  - Lagrangian duality (finding the best lower bound on the optimal value by solving simple “relaxations”)
- Primal/dual pair of LPs (“min” is traditionally viewed as the primal, “max” — as the dual)
- Dual of the dual is (equivalent to) the primal
- Duals of two equivalent LPs are themselves equivalent

Next

- Main Duality Theory results:
  - Weak duality and corollaries
  - Strong duality
  - Complementary slackness
- Additional interpretations and uses of the dual problem

Primal and Dual LPs

(P) \[\min_x \quad c^T x\]
\[\text{s.t.} \quad a^i x \geq b_i, \quad i \in M_1,\]
\[a^i x \leq b_i, \quad i \in M_2,\]
\[a^i x = b_i, \quad i \in M_3,\]
\[x_j \geq 0, \quad j \in N_1,\]
\[x_j \leq 0, \quad j \in N_2,\]
\[x_j \text{ free}, \quad j \in N_3,\]

(D) \[\max_p \quad p^T b\]
\[\text{s.t.} \quad p^i \geq 0, \quad i \in M_1\]
\[p^i \leq 0, \quad i \in M_2\]
\[p^i \text{ free}, \quad i \in M_3\]
\[p^T A_j \leq c_j, \quad j \in N_1\]
\[p^T A_j \geq c_j, \quad j \in N_2\]
\[p^T A_j = c_j, \quad j \in N_3\]

- If \( A \) is the constraint matrix, \( a^i \) is the \( i \)th row, and \( A_j \) is the \( j \)th column of \( A \)
- For theory (and in our lectures), by convention, the “min” problem is the “primal problem” and the “max” problem is the “dual problem”
- In practice, the original model is often referred to as the primal, and its dual is derived according to the above relationships (left to right, or right to left, as appropriate).
Weak duality

**Theorem 4.3 (Weak duality)**

If \( x \) is a feasible solution to the primal problem and \( p \) is a feasible solution to the dual problem, then

\[
p'b \leq c'x.
\]

**Proof:**

- Let \( u_i = p_i(a'_i x - b_i) \forall i \) and \( v_j = (c_j - p'_A_j)x_j \forall j \).
- Notice that (regardless of feasibility of \( x \) and \( p \))
  \[
  \sum_i u_i + \sum_j v_j = p'(Ax - b) + (c' - p'A)x = c'x - p'b
  \]
- If \( x \) and \( p \) are feasible, then
  \[
  u_i \geq 0 \forall i \in M_1 \cup M_2 \cup M_3 \text{ and } v_j \geq 0 \forall j \in N_1 \cup N_2 \cup N_3
  \]
- So, for feasible \( x \) and \( p \),
  \[
  c'x - p'b = \sum_i u_i + \sum_j v_j \geq 0.
  \]

Implications of weak duality

**Corollary 4.1**

(a) If the optimal cost in the primal is \(-\infty\), the the dual problem must be infeasible.

(b) If the optimal cost in the dual is \(+\infty\), the the primal problem must be infeasible.

**Proof:** by contradiction

(a) Suppose (P) is unbounded, but the dual has a feasible solution. Then its (finite) objective value provides a lower bound on \(-\infty\) — contradiction. (b) — similar proof.

**Corollary 4.2**

Let \( x \) and \( p \) be feasible solutions to the primal and the dual, respectively, and suppose that \( p'b = c'x \). Then \( x \) and \( p \) are optimal solutions to the primal and the dual, respectively.

**Proof:** If \( y \) is feasible for (P), then \( c'x = p'b \leq c'y \). Hence, \( x \) is optimal. Similar proof that \( p \) is optimal.
Strong Duality

**Theorem 4.4 (Strong duality)**

If a linear programming problem has an optimal solution, so does its dual, and the respective optimal costs are equal.

**Proof for a problem in standard form**

- Suppose the problem that has an optimal solution (which we refer to as (P), for convenience) is in standard form
  
  \[
  \begin{align*}
  (P) \quad \text{min} & \quad c^T x \\
  \text{s.t.} & \quad Ax = b, x \geq 0
  \end{align*}
  \]

  with linearly independent rows.

- Anticycling implementation of simplex finds an opt. basis \( B \):
  \[
  x_B = B^{-1} b \geq 0 \quad \text{and} \quad \bar{c}' = c' - c'_B B^{-1} A \geq 0.
  \]

- The dual LP has the form
  
  \[
  (D) \quad \text{max} \quad p^T b \quad \text{s.t.} \quad p^T A \leq c'
  \]

- Let \( p' = c'_B B^{-1} \). Then \( p \) is feasible for (D) and \( p^T b = c'x \).

- By Cor. 4.2, \( p \) is an opt. solution of (D).

To complete the proof:

- Now, consider the problem that has an optimal solution having a general form \( \Pi_1 \)

- Convert it into \( \Pi_2 \): an equivalent problem with the same optimal cost in standard form with lin. ind. rows

- According to the above proof, \( D_2 \) — the dual of \( \Pi_2 \) — has an optimal solution with the same opt. cost as \( \Pi_2 \).

- If \( D_1 \) and \( D_2 \) are duals of \( \Pi_1 \) and \( \Pi_2 \), then they are equivalent with the same optimal cost. (Thm. 4.2)

- Hence, \( D_1 \) has an optimal solution with the same optimal cost as \( \Pi_1 \).
Possibilities for primal/dual pairs:

<table>
<thead>
<tr>
<th>Finite optimum</th>
<th>Unbounded</th>
<th>Infeasible</th>
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<tbody>
<tr>
<td>Finite optimum</td>
<td>Possible</td>
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<tr>
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<tr>
<td>Infeasible</td>
<td>Impossible</td>
<td>Possible</td>
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Example of both problems infeasible:

\[
\begin{align*}
\text{min} & \quad x_1 + 2x_2 & \quad \text{max} & \quad p_1 + 3p_2 \\
\text{s.t.} & \quad x_1 + x_2 = 1 & \quad \text{s.t.} & \quad p_1 + 2p_2 = 1 \\
& \quad 2x_1 + 2x_2 = 3 & & \quad p_1 + 2p_2 = 2
\end{align*}
\]

– both infeasible.

Complementary Slackness

**Theorem 4.5 (Complementary slackness)**

Let \( x \) and \( p \) be feasible solutions to the primal and the dual problem, respectively. The vectors \( x \) and \( p \) are optimal solutions for the two respective problems if and only if:

\[
p_i(a'_ix - b_i) = 0, \quad \forall i, \quad \text{and} \quad (c_j - p'A_j)x_j = 0, \quad \forall j.
\]

**Proof:**

- Recall: \( u_i = p_i(a'_ix - b_i) \quad \forall i \) and \( v_j = (c_j - p'A_j)x_j \quad \forall j \)
- \( \sum_i u_i + \sum_j v_j = c'x - p'b \)
- When \( x \) and \( p \) are feasible, \( u_i \geq 0 \quad \forall i, \quad v_j \geq 0 \quad \forall j \)
- If \( x \) and \( p \) are optimal, then by strong duality,

\[
0 = c'x - p'b = \sum_i u_i + \sum_j v_j \geq 0,
\]

so \( u_i = 0 \quad \forall i \) and \( v_j = 0 \quad \forall j \).

- If \( u_i = 0 \quad \forall i \) and \( v_j = 0 \quad \forall j \), then \( c'x = p'b \), and Cor. 4.2 applies.
A “mechanical” analogy

- Consider a primal LP in the form \( \min \{ c'x : a_i'x \geq b_i, \forall i \} \), where the vector \( c \) is pointing upward.
- The optimum is achieved by letting a small ball loose in the feasible region — it will settle at lowest corner of polyhedron — point \( x^* \).
- At \( x^* \), the force of gravity is balanced by the forces exerted by the walls of the polyhedron, that is,
  \[
  c = \sum_i a_i p_i \quad \text{for some } p_i \geq 0 \quad \forall i
  \]
  so that vector \( p \) is a feasible solution for the dual problem.
- Note that force can only be exerted by the active constraints — hence,
  \[
  p_i(b_i - a_i'x^*) = 0 \quad \forall i,
  \]
  and so \( c'x^* = p'b \), and \( p \) is dual optimal.

Geometric view of duality

\[
\begin{align*}
\min & \quad c'x \\
\text{s.t.} & \quad a_i'x \geq b_i, \quad i = 1, \ldots, m
\end{align*}
\]
\[
\begin{align*}
\max & \quad p'b \\
\text{s.t.} & \quad \sum_{i=1}^m p_i a_i = c \\
& \quad p \geq 0
\end{align*}
\]

- Let \( x \) be a non-degenerate basic solution, and \( I \) be the unique index set of \( n \) linearly independent constraints active at \( x \).
- Consider a dual vector \( p \) that is complementary to \( x \) (but not necessarily dual feasible).
  - That is, \( p_i = 0, \quad i \notin I \), and so \( \sum_{i \in I} p_i a_i = c \).
  - Since \( a_i, \quad i \in I \) are linearly independent, \( p \) is determined by the above system of equation
    - i.e., \( p \) is a basic solution of the dual corresponding to \( I \).
- If \( x \) and \( p \) are as above, then
  - If \( x \) is feasible, then the **primal** basic solution corresponding to \( I \) is feasible.
  - If \( p \geq 0 \), then the **dual** basic solution corresponding to \( I \) is feasible.
  - If \( x \) and \( p \) are both feasible, then they are optimal.
Farkas' lemma

**Theorem 4.6 (Farkas' lemma)**

Let $A$ be a matrix of dimensions $m \times n$ and let $b$ be a vector in $\mathbb{R}^m$. Then, exactly one of the following two alternatives holds:

(a) There exists some $x \geq 0$ such that $Ax = b$.
(b) There exists some vector $p$ such that $p'A \geq 0'$ and $p'b < 0$.

- Example of theorems of alternatives
- $p$ as in (b) is a certificate of infeasibility of the system in (a), and vice versa

**Proof:**

- Easy to see that both (a) and (b) cannot hold: if $x$ is as in (a) and $p$ is as in (b), then

  $$0 > p'b = p'Ax \geq 0$$ — contradiction.

- Suppose (a) does not hold. Need to show that (b) does.

**Proof, continued:**

- Consider the pair of LPs

  $\begin{align*}
  \max & \quad 0'x \\
  \text{s.t.} & \quad Ax = b \\
  \min & \quad p'b \\
  \text{s.t.} & \quad p'A \geq 0' \\
  x & \geq 0
  \end{align*}$

- They are duals of each other
- Since (a) does not hold, the first LP is infeasible; hence the second is either infeasible or unbounded
- $0$ is a feasible solution of the second LP; thus it is unbounded
- Therefore, $p$ as in (b) exists.
Farkas’ lemma and asset pricing

- A single-period market with $n$ different assets
- $m$ possible “states of nature” at the end of the period.
- If we invest $1$ in asset $i$ now, and the state of nature turns out to be $s$, we receive a payoff of $r_{is}$
- Portfolio of assets: $x \in \mathbb{R}^n$, where $x_i$ — amount of asset $i$ held
  - $x_i < 0$ allowed: selling $|x_i|$ shares now, with a promise to buy $|x_i|$ shares of asset $i$ at the end of the period at a cost $r_{is}|x_i|
- Wealth resulting from $x$: $w_s = \sum_{i=1}^{n} r_{is}x_i$ is state $s$
  - In vector form: $w = Rx \in \mathbb{R}^m$
- The cost of acquiring portfolio $x$: $p'x$, where $p \in \mathbb{R}^n$ is the vector of asset prices.
- **Arbitrage:** existence of $x$ such that $Rx \geq 0$, while $p'x < 0$
  - Interpretation: arbitrage means free money now, no risks later
  - In efficient markets, prices $p$ should be such that arbitrage is absent. (A portfolio that carries no risks should be valuable.)
  - **Question:** what restriction on asset prices does the **absence of arbitrage** condition impose?

**Theorem 4.8**

The absence of arbitrage condition holds if and only if there exists a nonnegative vector $q = (q_1, \ldots, q_m)$, such that the price of each asset $i$ is given by

$$p_i = \sum_{s=1}^{m} q_s r_{si}.$$

**Proof:**

- The absence of arbitrage condition is: given $R$ and $p$, system $Rx \geq 0$, $p'x < 0$ has no solutions — compare to system in (b) of Farkas’ lemma
- By the lemma, above does not have a solution if and only if $p = R'q$, $q \geq 0$ has a solution
Farkas’ lemma and asset pricing

**Theorem 4.8**
The absence of arbitrage condition holds if and only if there exists a nonnegative vector \( \mathbf{q} = (q_1, \ldots, q_m) \), such that the price of each asset \( i \) is given by

\[
p_i = \sum_{s=1}^{m} q_s r_{si}.
\]

Interpretation:
- \( q_s \) is a “price of state of nature \( s \)”
  - A price of (imaginary) elementary asset that pays $1 if state \( s \) occurs, and $0 otherwise
- Theorem 4.8: In an efficient market with no arbitrage, asset prices must be consistent with these state of nature prices.
- State prices may not be unique unless \( \mathbf{R} \) satisfies a rank assumption