Recap, and outline of Lecture 12

\[ \min \{ c' x \mid Ax = b, \ x \geq 0 \} \]

\( B(1), \ldots, B(m) \) are the basic indices; \( B \) is the corresponding basis
\( N = \{ i \mid i \neq B(1), \ldots, B(m) \} \) are the non-basic indices
Basic dir. for \( j \in N: d \) with \( d_j = 1, \ d_B = -B^{-1}A_j, \ d_i = 0 \ o/w \)

Previously: Optimality conditions

- **Def. 3.3** A basis matrix \( B \) is optimal if
  - \( x_B = B^{-1}b \geq 0 \)
  - \( \bar{c}' = c' - c'_B B^{-1} A \geq 0 \)
- Suppose \( x \) is a BFS with basis \( B \), then
  - **Thm. 3.1(a)** If \( \bar{c} \geq 0 \), then \( x \) is an optimal (basic) feasible solution
  - If \( \bar{c}_j < 0 \) for some \( j \in N \) and \( d \) is the \( j \)th basic direction, then
    - \( d \) is an improving direction at \( x: c'x = \bar{c}_j < 0 \)
    - If \( x \) is non-degenerate, \( d \) is guaranteed to be a feasible direction at \( x \)

Today

- Development of the Simplex Method based on these ideas

Simplex method: Idea

Assume for now
- LP (in standard form) is feasible
- All its BFSs are non-degenerate
- We have found a starting BFS and corresponding basis

**Idea of the Simplex Method**

- Calculate \( \bar{c} \) at the current BFS
  - If \( \bar{c} \geq 0 \), stop — current basis is optimal (and hence current BFS is an optimal solution)
  - \( o/w \), pick a \( j \) with \( \bar{c}_j < 0 \); move along the \( j \)th basic direction until an adjacent BFS with a better cost is reached; repeat.

**Questions:**

1. How far do we move along a basic direction? Are we guaranteed to arrive at an adjacent BFS?
2. Will this algorithm terminate?
3. What if the LP is unbounded?
4. What about possible degeneracy?
5. How do we find the initial BFS? (And what if the LP is not feasible?)
How far to move in the basic direction: Questions 1 and 3

- Let \( \bar{c}_j < 0 \) and \( \mathbf{d} \) be the \( j \)th basic direction.
- Want to find \( \theta^* = \max \{ \theta \geq 0 \mid x + \theta \mathbf{d} \in P \} \)
- Since \( \mathbf{A} \mathbf{d} = 0 \), \( \mathbf{A}(x + \theta \mathbf{d}) = \mathbf{b} \) for any \( \theta \)
- Since \( \mathbf{d}_N \geq 0 \), \( x_N + \theta \mathbf{d}_N \geq 0 \) for any \( \theta \geq 0 \)
- Need: \( \theta^* = \max \{ \theta \geq 0 \mid x_B(i) + \theta d_B(i) \geq 0 \} \) for \( i = 1, \ldots, m \)
  - **Case 1:** \( d_B(i) \geq 0 \) for \( i = 1, \ldots, m \)
    - Then \( x + \theta \mathbf{d} \geq 0 \) for any \( \theta \geq 0 \), i.e., \( \theta^* = +\infty \)
    - In this case, the LP is unbounded
  - **Case 2:** \( d_B(i) < 0 \) for some \( i \)
    - \( \theta^* = \max \{ \theta \geq 0 \mid -\theta d_B(i) \leq x_B(i) \} \) for \( i : d_B(i) < 0 \), i.e.,
      
      \[
      \theta^* = \min_{ \{ i=1,\ldots, m \mid d_B(i) < 0 \} } \left( -\frac{x_B(i)}{d_B(i)} \right)
      \]
    - We move to point \( \mathbf{y} \) with \( y_j = \theta^* \), \( \mathbf{y}_B = \mathbf{x}_B + \theta^* \mathbf{d}_B \), \( y_i = 0 \ o/w \)

Moving to an adjacent BFS: Question 1 — continued

- Suppose we are in Case 2, and \( \theta^* = \min_{ \{ i \mid d_B(i) < 0 \} } \left( -\frac{x_B(i)}{d_B(i)} \right) \)
- Suppose \( l \) achieves the min.: \( d_B(l) < 0 \) and \( x_B(l) + \theta^* d_B(l) = 0 \)
- In non-degenerate case, basic variable \( x_B(l) \) became zero, and non-basic variable \( x_j \) became positive
  - \( x_B(l) \) should leave the basis, and \( x_j \) should enter the basis, replacing it, i.e.,
  - In the basis, column \( \mathbf{A}_{B(l)} \) should be replaced with \( \mathbf{A}_j \), giving
    
    \[
    \tilde{\mathbf{B}} = \begin{bmatrix}
    \mathbf{A}_{B(1)} & \cdots & \mathbf{A}_{B(l-1)} & \mathbf{A}_j & \mathbf{A}_{B(l+1)} & \cdots & \mathbf{A}_{B(m)}
    \end{bmatrix}
    \]

**Theorem 3.2**

(a) The columns \( \mathbf{A}_{B(i)}, i \neq l \), and \( \mathbf{A}_j \) are linearly independent and, therefore, \( \tilde{\mathbf{B}} \) is a basis matrix.

(b) The vector \( \mathbf{y} = \mathbf{x} + \theta^* \mathbf{d} \) is a basic feasible solution associated with the basis matrix \( \tilde{\mathbf{B}} \).
Proof of Theorem 3.2

Part (a)

- If the columns of $\bar{B}$ are not lin.ind., then $\exists$ nontrivial lin.comb.

$$0 = \sum_{i=1,\ldots,m} \lambda_i A_{\bar{B}(i)} = \sum_{i=1,\ldots,l-1,l+1,\ldots,m} \lambda_i A_{\bar{B}(i)} + \lambda_j A_j$$

- Pre-multiplying by $B^{-1}$, this implies:

$$0 = \sum_{i \neq l} \lambda_i B^{-1} A_{\bar{B}(i)} + \lambda_j B^{-1} A_j = \sum_{i \neq l} \lambda_i e^i - \lambda_j d_B$$

- Thus, $\lambda_l = 0$ (since $d_{B(l)} < 0$ and $e^i_l = 0$ for $i \neq l$)
- This would imply that $A_{\bar{B}(i)}, i \neq l$ are linearly dependent — contradiction!

Part (b)

- $y \geq 0$ and $Ay = b$
- $y_i = 0$ for $i \neq \bar{B}(1), \ldots, \bar{B}(m)$
- Columns $A_{\bar{B}(1)}, \ldots, A_{\bar{B}(m)}$ are lin.ind. — see part (a)
- Thus $y$ is a BFS associated with the basis $\bar{B}$.

An iteration of the simplex method (a “pivot”)

1. Iteration starts with a basis of columns $A_{\bar{B}(1)}, \ldots, A_{\bar{B}(m)}$, and an associated BFS $x$.
2. Compute the reduced costs $\bar{c}_j = c_j - c_j^T B^{-1} A_j$ for all nonbasic indices $j$. If $\bar{c}_j \geq 0$ for all $j$, terminate with an optimal solution. Otherwise, pick some $j$ with $\bar{c}_j < 0$.
3. Compute $u = B^{-1} A_j$. If $u \leq 0$, we have $\theta^* = \infty$ and the problem is unbounded. (Note: $u = -d_B \in \mathbb{R}^m$.)
4. If some component of $u$ is positive, let

$$\theta^* = \min_{i: u_i > 0} \frac{x_{B(i)}}{u_i}.$$ 

5. Let $l$ be such that $\theta^* = x_{B(l)}/u_l$. Form a new basis by replacing $A_{\bar{B}(l)}$ with $A_j$. If $y$ is the new basic feasible solution, the values of the new basic variables are $y_j = \theta^*$ and $y_{B(i)} = x_{B(i)} - \theta^* u_i$, $i \neq l$. 