Recap, and outline of Lecture 10

\[ \begin{align*}
\text{min} & \quad c'x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0
\end{align*} \]

Previously
- Converting any LP into standard form (✓)
- Interpretation and visualization (✓)
- Full row rank assumption on A (✓)
- Basic solutions and bases in standard form polyhedra — started

Today
- Basic solutions and bases in standard form polyhedra
- Degeneracy in standard form polyhedra
- Adjacent solutions and adjacent bases
- Optimality conditions (for standard form LPs)

Basic solutions in standard form polyhedra

\[ Ax = b, \; x \geq 0 \]

Assumptions
- \( A \in \mathbb{R}^{m \times n}, \; b \in \mathbb{R}^m \)
- \( Ax = b \) has at least one solution
- \( A \) has full row rank (and hence \( m \leq n \))

Recall:
- To construct a basic solution, need to choose \( n \) linearly independent constraints to be active.
- At a basic solution for a problem in standard form:
  - \( Ax = b \) give us \( m \) linearly independent active constraints
  - (At least) \( n - m \) of the constraints “\( x_j \geq 0 \)” need to be active
  - The resulting system of linear equations needs to have a unique solution!
Example

Arbitrarily picking \( n - m \) sign constraints to be active might not result in a basic solution!

\[
A = \begin{bmatrix}
1 & 1 & 2 & 1 & 0 & 0 & 0 \\
2 & 1 & 6 & 0 & 1 & 0 & 0 \\
1 & 0 & 4 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad b = \begin{bmatrix}
8 \\
12 \\
4 \\
6
\end{bmatrix}
\]

\( n = 7, \; m = 4 \); need (at least) 3 sign constraints active for a BS

1. Try \( x_1 = x_2 = x_3 = 0 \) (a BFS)
2. Try \( x_1 = x_2 = x_4 = 0 \) (a BS, but not a BFS)
3. Try \( x_1 = x_3 = x_6 = 0 \) (not a BS: no solutions)
4. Try \( x_4 = x_5 = x_6 = 0 \) (not a BS: multiple solutions, e.g.,
\[
x_1 = 2, \; x_2 = 5, \; x_3 = 0.5, \; x_7 = 1, \text{ or}
\]
\[
x_1 = 1, \; x_2 = 5.5, \; x_3 = 0.75, \; x_7 = 0.5
\]

Basic solutions in standard form polyhedra

**Theorem 2.4**

Consider the polyhedron represented by constraints \( Ax = b \) and \( x \geq 0 \) and assume that the \( m \times n \) matrix \( A \) has linearly independent rows. A vector \( x \in \mathbb{R}^n \) is a basic solution if and only if we have \( Ax = b \) and there exist indices \( B(1), \ldots, B(m) \) such that:

- (a) The columns \( A_{B(1)}, \ldots, A_{B(m)} \) are linearly independent;
- (b) if \( j \neq B(1), \ldots, B(m) \), then \( x_j = 0 \).

**Proof of the “if” part:**

- Suppose \( x \) satisfies (a) and (b). Then \( x \) satisfies

\[
\sum_{i=1}^{m} A_{B(i)} x_{B(i)} = b, \quad x_j = 0, \; j \neq B(1), \ldots, B(m)
\]

- Above system has a unique solution (since \( A_{B(1)}, \ldots, A_{B(m)} \) are linearly independent)
- Therefore, \( x \) is a BS
Basic solutions in standard form polyhedra

**Theorem 2.4**
Consider the polyhedron represented by constraints $Ax = b$ and $x \geq 0$ and assume that the $m \times n$ matrix $A$ has linearly independent rows. A vector $x \in \mathbb{R}^n$ is a basic solution if and only if we have $Ax = b$ and there exist indices $B(1), \ldots, B(m)$ such that:
(a) The columns $A_{B(1)}, \ldots, A_{B(m)}$ are linearly independent;
(b) if $j \neq B(1), \ldots, B(m)$, then $x_j = 0$.

**Proof of the “only if” part:**
- Suppose $x$ is a BS.
- Let $x_{B(1)}, \ldots, x_{B(k)}$ be the non-zero components of $x$ ($k \leq m$)
- The following system has a unique solution (since $x$ is a BS):
  \[ \sum_{i=1}^{k} A_{B(i)}x_{B(i)} = b, \; x_j = 0, \; j \neq B(1), \ldots, B(k) \]
- Hence, $A_{B(1)}, \ldots, A_{B(k)}$ are linearly independent
- If $k < m$, can find additional columns $A_{B(k+1)}, \ldots, A_{B(m)}$ so that columns $A_{B(1)}, \ldots, A_{B(m)}$ are linearly independent
- With this selection of $B(1), \ldots, B(m)$, $x$ satisfies (a) and (b)

Procedure for constructing basic solutions of problems in standard form

1. Choose $m$ linearly independent columns $A_{B(1)}, \ldots, A_{B(m)}$
2. Let $x_i = 0$ for all $i \neq B(1), \ldots, B(m)$
3. Solve the system of $m$ equations $Ax = b$ for the unknowns $x_{B(1)}, \ldots, x_{B(m)}$

**Example:**

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 6 & 0 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; b = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}$$

- Let $B(1) = 4, \; B(2) = 1, \; B(3) = 6, \; B(4) = 2$.
- $x_3 = x_5 = x_7 = 0$.
- Solve

$$x_{B(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; x_{B(2)} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \; x_{B(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; x_{B(4)} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$x_4 = x_{B(1)} = -1, \; x_1 = x_{B(2)} = 3, \; x_6 = x_{B(3)} = 1, \; x_2 = x_{B(4)} = 6$$
**Terminology of BSs for standard form systems**

If $x$ is a basic solution, and $x_{B(1)}, \ldots, x_{B(m)}$ are as above,

- Columns $A_{B(1)}, \ldots, A_{B(m)}$ — *basic columns*; they form a basis of $\mathbb{R}^m$.

- The matrix $B = \begin{bmatrix} A_{B(1)} & A_{B(2)} & \cdots & A_{B(m)} \end{bmatrix}$ is the *basis matrix*.

- Variables $x_B = (x_{B(1)}, \ldots, x_{B(m)})'$ — *basic variables*; the remaining variables are *nonbasic*.

- Unique solution of $Bx_B = b$ is $x_B = B^{-1}b$.

- A BFS “synthesizes” the target vector $b$ as a (nonnegative) linear combination of basic columns of $A$.

**Degeneracy in standard form polyhedra**

**Definition 2.10**

A basic solution $x \in \mathbb{R}^n$ is said to be *degenerate* if more than $n$ of the constraints are active at $x$.

**Definition 2.11**

Consider the standard from polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b, \ x \geq 0\}$ and let $x$ be a basic solution. Let $A \in \mathbb{R}^{m \times n}$ have full row rank. The vector $x$ is a *degenerate* basic solution if more than $n - m$ of the components of $x$ are zero.

Degeneracy is not a purely geometric property! It depends on problem representation.

$P = \{(x_1, x_2, x_3) : x_1 - x_2 = 0, \ x_1 + x_2 + 2x_3 = 2, \ x_1, x_2, x_3 \geq 0\}$

vs.

$P = \{(x_1, x_2, x_3) : x_1 - x_2 = 0, \ x_1 + x_2 + 2x_3 = 2, \ x_1, x_3 \geq 0\}$

$(0, 0, 1)$ is degenerate in the 1st representation, but not the 2nd.
Correspondence between bases and basic solutions

- Two bases are **distinct** or **different** if they involve different set of indices \( \{B(1), \ldots, B(m)\} \).
- Two different bases may correspond to the same basic solution.
  - In the second part of Theorem 2.4, if the number of non-zero components in a BS is \( < m \) (i.e., the basic solution is **degenerate**), we might have a choice of which columns of \( A \) to use to complete the basis.
- However, different basic solutions correspond to different bases
  - because a basis uniquely determines the corresponding basic solution

Adjacent basic solutions and adjacent bases

**Definition: Adjacent basic solutions**

Two distinct basic solution to a set of linear constraints in \( \mathbb{R}^n \) are adjacent if we can find \( n - 1 \) linearly independent constraints that are active at both of them.

**Definition: Adjacent bases**

In a standard form problem, two bases are adjacent if they share all but one basic column.

- Adjacent basic solutions can always be obtained from adjacent bases
- If two adjacent bases lead to two different basic solutions, then these solutions are adjacent.
Towards an algorithm for solving LPs

**Typical algorithm for solving an optimization problem:**

- Find a feasible solution
- Check if there is a nearby feasible solution with a lower cost.
  - If so, move to that solution and repeat. If no — terminate;
    the current solution is a *locally optimal solution*
    - i.e., feasible solution with the lowest cost among nearby feasible solutions

**Observations**

- In linear programming, a locally optimal solution is also *globally optimal*
- Need an approach to determine a direction of cost decrease
  - ...or establish that none exists

**Definition 3.1**

Let $x$ be an element of a polyhedron $P$. A vector $d \in \mathbb{R}^n$ is said to be a *feasible direction* at $x$, if there exists a positive scalar $\theta$ for which $x + \theta d \in P$.

**Basic directions**

- Let $x$ be a BFS, with $B(1), \ldots, B(m)$ being the indices of the basic variables
- $x_i = 0$ for every non-basic variable
- $B = [A_{B(1)}, \ldots, A_{B(m)}]$ — corresponding basis matrix
- $x_B = (x_{B(1)}, \ldots, x_{B(m)})' = B^{-1}b$.

**The $j$th basic direction:**

- Let $j \neq B(1), \ldots, B(m)$
- Idea: move away from $x$ by setting $x_j = \theta > 0$, and keeping all other non-basic variables at 0.
- Formally: move from $x$ to $x + \theta d$, where
  - $d_j = 1$
  - $d_i = 0$ for all $i \neq j, B(1), \ldots, B(m)$
  - Need $A(x + \theta d) = b$, i.e., $Ad = 0$
    - $0 = Ad = \sum_{i=1}^{n} A_i d_i = \sum_{i=1}^{m} A_{B(i)} d_{B(i)} + A_j = Bd_B + A_j$
    - $d_B = -B^{-1}A_j$