Worksheet Juxtaposition

1. Consider a mirror in the shape of the graph of $y = \pm \sqrt{4x}$.
   
   (a) Draw the mirror (make it big). What shape is it?
   
   (b) Draw a light ray travelling leftward along the line $y = -b$, where $b$ is some positive number (making $-b$ negative). At what point $P$ does the ray hit the mirror?
   
   (c) Find, in terms of $b$, the slope of the tangent to the mirror at $P$.
   
   (d) The normal to a curve at a point is the line through that point which is perpendicular to the tangent line. Find the slope of the normal to the mirror at $P$, and draw both the normal and tangent lines on your graph.
   
   (e) Suppose a line makes an angle $\theta$ with the $x$-axis. What is the slope of the line?
   
   (f) Let $\theta$ be the angle the normal to the mirror at $P$ makes with the light ray $y = -b$. Can you write $\theta$ in terms of $b$? Hint: Use (1d) and (1e).

To be continued...

2. Suppose you are asked to design the first ascent and drop for a new roller coaster at Cedar Point. (You get to name it, too!) By studying photographs of your favorite coasters, you decide to make the slope of the ascent 0.8 and the slope of the drop $-1.6$. You decide to connect these two straight inclines $y = L_1(x)$ and $y = L_2(x)$ with part of a parabola $y = f(x) = ax^2 + bx + c$, where $x$ and $f(x)$ are measured in feet. For the track to be smooth there can’t be abrupt changes in direction, so you want the linear segments $L_1$ and $L_2$ to be tangent to the parabola at the transition points $P$ and $Q$. To simplify the equations, you decide to put the origin at $P$.

   (a) Name your coaster.
   
   (b) Suppose the horizontal distance between $P$ and $Q$ is 100 feet. Write equations in $a$, $b$, and $c$ that will ensure that the track is smooth at the transition points.
   
   (c) Solve the equations in (2b) for $a$, $b$, and $c$ to find a formula for $f(x)$.
   
   (d) Plot $L_1$, $f$, and $L_2$ to verify graphically that the transitions are smooth.
   
   (e) Find the difference in elevation between $P$ and $Q$.
   
   (f) Suppose the base of the hill (the distance from $A$ to $B$ in the picture) is 300 feet long. How high is the hill?
3. (This problem appeared on a Winter 2007 Math 115 exam) Suppose \( f \) and \( g \) are differentiable functions with values given by the table below.

(a) If \( h(x) = f(x)g(x) \), find \( h'(3) \).

(b) If \( j(x) = \frac{(g(x))^3}{f(x)} \), find \( j'(1) \).

(c) If \( d(x) = x \ln \left(e^{f(x)}\right) \), find \( d'(3) \).

(d) If \( t(x) = \cos(g(x)) \), find \( t'(1) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
<th>( f'(x) )</th>
<th>( g'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>9</td>
<td>-3</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>11</td>
<td>15</td>
<td>-19</td>
</tr>
</tbody>
</table>

4. Let \( f(x) = x^2 - 2x + 4 \) and \( g(x) = -x^2 - 2x - 10 \).

(a) Draw \( y = f(x) \) and \( y = g(x) \) on the same set of axes. How many lines are tangent to both graphs?

(b) Find the equation of a line with slope \( m \) that is tangent to the graph of \( f \). (So, your answer should be in terms of \( m \).)

(c) Do the same for \( g \).

(d) Find the equations of the lines which are tangent to both graphs.

5. Let’s see if we can prove that the derivative of \( \sin(x) \) is \( \cos(x) \). Remember that last time we showed that

\[
\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1.
\]

(a) Show that

\[
\lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta} = 0.
\]

Hint: Multiply the top and bottom by \( 1 + \cos(\theta) \), and simplify.

(b) Write down the definition of the derivative of \( \sin(x) \) at \( x = a \).

(c) Use a trig identity to write \( \sin(a + h) \) in terms of sines and cosines of \( a \) and \( h \).

(d) Now use the two limits we know (the one from last time and the one in part (a)) to simplify the derivative.