1. This exam has 16 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.

2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.

3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.

4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.

5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a 3” × 5” note card.

6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.

7. You must use the methods learned in this course to solve all problems.
<table>
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<tr>
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<th>Exam</th>
<th>Problem</th>
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Recommended time (based on points): 180 minutes
7. You decide to take a weekend off and drive down to Chicago. The graph below represents your distance $S$ from Ann Arbor, measured in miles, $t$ hours after you set out.

Let $A(t)$ be the slope of the line connecting the origin $(0,0)$ to the point $(t, S(t))$.

(a) (3 points) What does $A(t)$ represent in everyday language?

$A(t) = \frac{S(t)}{t}$ represents your average velocity, in miles per hour, during the first $t$ hours of the trip.

(b) (3 points) Estimate the time $t$ at which $A(t)$ is maximized. Write a one sentence explanation and use the graph above to justify your estimate.

$A(t)$ is maximized when the line connecting the origin to the curve is steepest. By inspecting the graph, it is clear that this happens at around $t = 3$ hours, at which point the line is tangent to the curve. (This line is indicated in the figure by the dashed line.)

(c) (4 points) Use calculus to explain why $A(t)$ has a critical point when the line connecting the origin to the point $(t, S(t))$ is tangent to the curve $S(t)$.

By applying the quotient rule, we find that

$$A'(t) = \frac{t \cdot S'(t) - S(t)}{t^2}.$$ 

At any non-zero value of $t$ for which the line through the origin to $(t, S(t))$ is tangent to the curve, we must have $S'(t)$ (the slope of the tangent line) equal to $S(t)/t$ (the slope of the line from the origin to $(t, S(t))$); thus, for any such $t$, $S(t) = t \cdot S'(t)$. Thus, for any such $t$, $A'(t) = 0$, i.e. $A(t)$ has a critical point there.
1. [12 points] Consider the graph of \( j'(x) \) given here. Note that this is not the graph of \( j(x) \).

For each of (a)-(f) below, list all \( x \)-values labeled on the graph which satisfy the given statement in the blank provided. If the statement is not true at any of the labeled \( x \)-values, write “NP”. You do not need to show your work. No partial credit will be given on each part of this problem.

(a) The function \( j(x) \) has a local minimum at \( x = \boxed{C} \).
(b) The function \( j(x) \) has a local maximum at \( x = \boxed{A, E} \).
(c) The function \( j(x) \) is concave up at \( x = \boxed{B, C, D} \).
(d) The function \( j(x) \) is concave down at \( x = \boxed{A, E} \).
(e) The function \( j'(x) \) has a critical point at \( x = \boxed{NP} \).
(f) The function \( j''(x) \) is greatest at \( x = \boxed{C} \).
7. [14 points] For positive $A$ and $B$, the force between two atoms is a function of the distance, $r$, between them:

$$f(r) = -\frac{A}{r^2} + \frac{B}{r^3} \quad r > 0.$$

a. [2 points] Find the zeroes of $f$ (in terms of $A$ and $B$).

**Solution:** Finding a common denominator for $f$, we have

$$f(r) = -\frac{Ar + B}{r^3}.$$  

This means $f(r) = 0$ when the numerator is zero, so $r = \frac{B}{A}$ is the only zero of $f$.

b. [7 points] Find the coordinates of the critical points and inflection points of $f$ in terms of $A$ and $B$.

**Solution:** Seeking critical points, we take the derivative of $f(r)$ and set it equal to zero

$$f'(r) = 2\frac{A}{r^3} - 3\frac{B}{r^4} = \frac{2Ar - 3B}{r^4} = 0.$$  

Solving, we have that $r = \frac{3B}{2A}$ is our only critical point.

Now seeking inflection points, we take the second derivative of $f(r)$ and set it equal to zero.

$$f''(r) = -\frac{6A}{r^4} + \frac{12B}{r^5} = \frac{12B - 6Ar}{r^5} = 0.$$  

Solving, we have that $r = \frac{2B}{A}$ is a candidate for an inflection point. Now we must test to see whether this is an inflection point. We compute $f''(\frac{2B}{A}) = \frac{6B}{(B/A)^2} > 0$ since $A$ and $B$ are both positive, and also, $f''\left(\frac{3B}{2A}\right) = \frac{-6B}{(3B/2A)^2} < 0$ since $A$ and $B$ are both positive. This means $f''$ changes sign from positive to negative across the point $r = \frac{2B}{A}$, so it must be an inflection point.

c. [5 points] If $f$ has a local minimum at $(1, -2)$ find the values of $A$ and $B$. Using your values for $A$ and $B$, justify that $(1, -2)$ is a local minimum.

**Solution:** We already know our only critical point is $r = \frac{3B}{2A}$. If $f$ has a local minimum at $(1, -2)$, we must have that $1 = r = \frac{3B}{2A}$, so that $2A = 3B$. In addition, $-2 = f(1) = -A + B$. Solving these equations simultaneously, we have $A = 6$ and $B = 4$. We have already computed

$$f''(r) = \frac{12B - 6Ar}{r^5} = \frac{48 - 36r}{r^5}.$$  

So $f''(1) = 48 - 36 = 12 > 0$ which means the critical point $(1, -2)$ is a local minimum since $f$ is concave up at this point.
7. (12 points) The flux $F$, in millilitres per second, measures how fast blood flows along a blood vessel. Poiseuille’s Law states that the flux is proportional to the fourth power of the radius, $R$, of the blood vessel, measured in millimeters. In other words $F = kR^4$ for some positive constant $k$.

(a) Find a linear approximation for $F$ as a function of $R$ near $R = 0.5$. (Leave your answer in terms of $k$).

The linear approximation near $R = 0.5$ is given by $F(R) \approx F(0.5) + F'(0.5)(R - 0.5)$.

\[
F(0.5) = k(0.5)^4 = \frac{k}{16},
\]
\[
F'(R) = 4kR^3,
\]
\[
F'(0.5) = 4k(0.5)^3 = \frac{k}{2}
\]

Thus for $R$ near 0.5, $F(R)$ is given by:

\[
F(R) \approx \frac{k}{16} + \frac{k}{2}(R - 0.5)
\]

(b) A partially clogged artery can be expanded by an operation called an angioplasty, which widens the artery to increase the flow of blood. If the initial radius of the artery was 0.5mm, use your approximation from part (a) to approximate the flux when the radius is increased by 0.1mm.

To approximate when the radius increases by 0.1mm from 0.5mm we can evaluate the linear approximation from (a) at 0.6 giving:

\[
F(0.6) \approx \frac{k}{16} + \frac{k}{2}(0.6 - 0.5),
\]
\[
= \frac{k}{16} + \frac{k}{20},
\]
\[
= \frac{9k}{80} = (0.1125)k \text{ millilitres per second}.
\]

(c) Is the answer found in part (b) an under- or over-approximation? Justify your answer.

In order to determine if this is an over or an under approximation we use the second derivative:

\[
F''(R) = 12kR^2
\]

Since $k > 0$, $F''(R)$ is always positive, and the function is always concave up. Thus the linear approximation will always be an under approximation.
6. [15 points] Given below is the graph of a function \( h(t) \). Suppose \( j(t) \) is the local linearization of \( h(t) \) at \( t = \frac{7}{8} \).

\[ 
\begin{array}{c}
\text{ Graph of } h(t) \\
\end{array}
\]

\( y \)

\( t \)

\( (\frac{7}{8}, \frac{1}{4}) \)

\[ 
\begin{align*}
\text{a. } & \text{[5 points] Given that } h'(\frac{7}{8}) = \frac{2}{3}, \text{ find an expression for } j(t). \\
& \text{Solution: The local linearization is the tangent line to the curve. We know this line has} \\
& \text{slope } h'(\frac{7}{8}) = \frac{2}{3} \text{ and it goes through the point } (\frac{7}{8}, \frac{1}{4}), \text{ so it has equation} \\
& y - \frac{1}{4} = \frac{2}{3}(t - \frac{7}{8}) \\
& \text{using point slope form. Solving for } y \text{ we have } y = \frac{2}{3}t - \frac{1}{3}. \text{ So } j(t) = \frac{2}{3}t - \frac{1}{3}. \\
\end{align*}
\]

\[ 
\begin{align*}
\text{b. } & \text{[4 points] Use your answer from (a) to approximate } h(1). \\
& \text{Solution: Since } j(t) \text{ approximates } h(t) \text{ for } t\text{-values near } \frac{7}{8}, \text{ we have} \\
& h(1) \approx j(1) = \frac{2}{3}(1) - \frac{1}{3} = \frac{1}{3}. \\
\end{align*}
\]

\[ 
\begin{align*}
\text{c. } & \text{[3 points] Is the approximation from (b) an over- or under-estimate? Explain.} \\
& \text{Solution: The approximation in (b) is an underestimate. The function } h(t) \text{ is concave} \\
& \text{up at } t = 7/8 \text{ which means the graph lies above the local linearization for } t\text{-values near} \\
& 7/8. \text{ Since we are using the local linearization to estimate the function value, our estimate} \\
& \text{will be less than the actual function value.} \\
\end{align*}
\]

\[ 
\begin{align*}
\text{d. } & \text{[3 points] Using } j(t) \text{ to estimate values of } h(t), \text{ will the estimate be more accurate at} \\
& t = 1 \text{ or at } t = \frac{3}{4}? \text{ Explain.} \\
& \text{Solution: The estimate at } t = 3/4 \text{ will be more accurate. This can be seen by drawing} \\
& \text{the tangent line and measuring the vertical distance between the estimated value and} \\
& \text{the function value at the } t \text{ values 3/4 and 1. The line is much closer to the function at} \\
& t = 3/4 \text{ than it is at } t = 1. \\
\end{align*}
\]
8. [12 points] In the following table, both $f$ and $g$ are differentiable functions of $x$. In addition, $g(x)$ is an invertible function. Write your answers in the blanks provided. You do not need to show your work.

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</table>

a. [3 points] If $h(x) = \frac{g(x)}{f(x)}$, find $h'(4)$.

$$h'(4) = \frac{-15}{4}$$

b. [3 points] If $k(x) = f(x)g(x)$, find $k'(2)$.

$$k'(2) = 5$$

c. [3 points] If $m(x) = g^{-1}(x)$, find $m'(4)$.

$$m'(4) = \frac{1}{2}$$

d. [3 points] If $n(x) = f(g(x))$, find $n'(3)$.

$$n'(3) = 6$$
4. (6 points) The shape of a balloon used by a clown for making a balloon animal can be approximated by a cylinder. As the balloon is inflated, assume that the radius is increasing by 2 cm/sec and the height is given by \( h = 2r \). At what rate is air being blown into the balloon at the moment when the radius is 3 cm?

The formula for the volume of a cylinder with radius \( r \) and height \( h \) is given by \( V = \pi r^2 h \). We know that \( h = 2r \), so we can write \( V = 2\pi r^3 \). Taking the derivative with respect to \( t \) of both sides we get

\[
\frac{dV}{dt} = 2\pi 3r^2 \frac{dr}{dt}.
\]

We are interested at the time when \( r = 3 \) and \( \frac{dr}{dt} = 2 \), so

\[
\frac{dV}{dt} = 108\pi \text{cm}^3/\text{sec}.
\]

5. (8 points) In introductory physics one learns the formula \( F = ma \), connecting the force on an object, \( F \), with the mass of the object and the acceleration that the object experiences under the force. One also learns the formula \( p = mv \) where \( p \) is the momentum of an object, \( m \) is the mass, and \( v \) is the velocity.

(a) Derive the formula \( F = ma \) given that \( \frac{dp}{dt} = F \), assuming that the mass is constant and that \( p = mv \). Explain your answer.

Take the derivative of \( p = mv \) with respect to \( t \) to get

\[
\frac{dp}{dt} = m \frac{dv}{dt}
\]

but since acceleration is the derivative of velocity, this gives

\[
F = ma.
\]

(b) Derive a formula for the force \( F \) if the mass is not assumed to be constant.

We do the same thing as in part (a), except this time \( mv \) is a product of two functions of \( t \). Therefore we get

\[
F = \frac{dp}{dt} = v \frac{dm}{dt} + ma.
\]
8. [12 points] The equation \((x^2 + y^2)^2 = 4x^2y\) describes a two-petaled rose curve.

a. [2 points] Verify that the point \((x, y) = (1, 1)\) is on the curve.

Solution: At the point \((x, y) = (1, 1)\),
\[
(x^2 + y^2)^2 = (1^2 + 1^2)^2 = 4 = 4(1)^2(1) = 4x^2y.
\]

b. [7 points] Calculate \(dy/dx\) at \((x, y) = (1, 1)\).

Solution: Differentiating both sides of the equation for the curve with respect to \(x\) we have
\[
2(x^2 + y^2) \left(2x + 2y \frac{dy}{dx}\right) = 4 \left(2xy + x^2 \frac{dy}{dx}\right).
\]
At the point \((x, y) = (1, 1)\) this equation becomes
\[
2(1^2 + 1^2) \left(2(1) + 2(1) \frac{dy}{dx}\right) = 4 \left(2(1)(1) + (1)^2 \frac{dy}{dx}\right).
\]
Simplifying, we have \(4(2 + 2 \frac{dy}{dx}) = 8 + 4 \frac{dy}{dx}\). This gives us that \(\frac{dy}{dx} = 0\) at \((x, y) = (1, 1)\).

c. [3 points] Find the equation of the tangent line to the rose curve at the point \((x, y) = (1, 1)\).

Solution: Using point slope form, the tangent line is \(y - 1 = 0(x - 1)\). Simplifying, we have that the tangent line to the rose curve at \((x, y) = (1, 1)\) is \(y = 1\).
7. [10 points] For each real number $k$, there is a curve in the plane given by the equation \( e^{y^2} = x^3 + k \).

a. [4 points] Find \( \frac{dy}{dx} \).

Solution: We have \( 2y e^{y^2} \frac{dy}{dx} = 3x^2 \), so
\[
\frac{dy}{dx} = \frac{3x^2}{2y e^{y^2}}
\]

b. [3 points] Suppose that $k = 9$. There are two points on the curve where the tangent line is horizontal. Find the $x$ and $y$ coordinates of each one.

Solution: Horizontal tangent lines occur when the numerator of the derivative is zero, so in this case $x = 0$. To solve for the $y$-coordinate, we have \( e^{y^2} = 9 \) so \( y = \pm \sqrt{\ln(9)} \).

c. [3 points] Now suppose that $k = \frac{1}{2}$. How many points are there where the curve has a horizontal tangent line?

Solution: Again we get $x = 0$. Now if we try to solve for $y$ we have \( y^2 = \ln \left( \frac{1}{2} \right) < 0 \) and so there are no points where the curve has a horizontal tangent line.
7. (17 points) At the Wizard Fair, there is a booth where wizards win Bertie Bott’s Every Flavor Beans. To determine how many beans one gets, a contestant is given a string 50 inches long. From this string, contestants can cut lengths to form an equilateral triangle and a rectangle whose length is twice its width. The number of Bertie Bott’s beans one wins depends on the combined areas of the triangle and rectangle. Harry, knowing calculus, goes immediately to work setting up a function, finding critical points, etc.

(a) Use your knowledge of calculus to determine the areas of the triangle and rectangle that will maximize the number of beans that Harry can win. Show your work.

We need to find a formula for the total area of the triangle and rectangle. Let’s begin by finding a formula for the area of the triangle. Say we cut the string $x$ inches from the left end of the string, and suppose the left piece is used to make the equilateral triangle and the right piece is used to make the rectangle. So, each side of the triangle must have length $\frac{x}{3}$. An equilateral triangle has all of its angles equal to $\frac{\pi}{3}$. Let $h$ be the height if the triangle. Then $\sin(\frac{\pi}{3}) = \frac{h}{\frac{x}{3}}$ and $h = \frac{x\sqrt{3}}{6}$.

So area of triangle = $\frac{1}{2}\left(\frac{x}{3}\right)\left(\frac{x\sqrt{3}}{6}\right)$.

Now we need to find a formula for the area of the rectangle. Let the two sides of the rectangle be $l$ and $w$. Then the perimeter of the rectangle must equal the length of the right piece of string. So, $2l + 2w = 50 - x$. We know though that the length is twice the width, which means $2(2w) + 2w = 6w = 50 - x$, and therefore, $w = \frac{50 - x}{6}$. Since the area of the rectangle equals $lw$, we know that the area equals $2w^2 = 2\left(\frac{50 - x}{6}\right)^2$.

Thus we have $A_{total} = \frac{x^2\sqrt{3}}{36} + \frac{2(50 - x)^2}{36}$

$$= \frac{1}{36}\left((\sqrt{3} + 2)^2 - 200x + 2(50)^2\right).$$

So $\frac{dA}{dt} = \frac{1}{36}(2(\sqrt{3} + 2)x - 200)$. Setting the derivative equal to zero to find the critical point gives $0 = (\sqrt{3} + 2)x - 100$ and thus $x \approx 26.79$. Notice though that $\frac{d^2A}{dt^2} = \frac{1}{18}(\sqrt{3} + 2) > 0$, which means that $A$ has a minimum at $x = 26.79$ and the maximum must occur at one of the endpoints - i.e. when $x = 0$ or $x = 50$.

$x = 0$: $2w + 2l = 6w = 50 \Rightarrow w = 8.34$. Thus, $A_{total} = lw = 2(8.34)^2 = 136.89$ square inches.

$x = 50$: area $= \frac{1}{2}\left(\frac{50}{3}\right)(\frac{50\sqrt{3}}{6}) \approx 120.28$ square inches.

This analysis says that Harry should not cut the string and the maximum occurs when the area of the triangle is zero and the area of the rectangle is 136.89 square inches.

(b) If the number of beans won is 9 times the combined area, what is the greatest number of beans a contestant can win?

This is just nine times the area we found in (a). So, the greatest number of beans a contestant can win is $9 \times 136.89 = 1232$ beans.
5. (16 points) A directional microphone is mounted on a stand facing a wall. The sensitivity $S$ of the microphone to sounds at point $X$ on the wall is inversely proportional to the square of the distance $d$ from the point $X$ to the mic, and directly proportional to the cosine of the angle $\theta$. That is, $S = K \frac{\cos \theta}{d^2}$ for some constant $K$. (See the diagram below.) How far from the wall should the mic be placed to maximize sensitivity to sounds at $X$?

![Diagram of a microphone facing a wall with labels $d$, $w$, and $15$ ft.]

We are given that $S = K \frac{\cos \theta}{d^2}$. Also, from the definition of cosine we see $\cos \theta = \frac{w}{d}$. By the Pythagorean theorem, $d^2 = w^2 + 15^2 = w^2 + 225$. Therefore

$$S = K \frac{\cos \theta}{d^2} = K \frac{\frac{w}{d}}{w^2 + 225} = K \frac{w}{(w^2 + 225)^{3/2}}.$$

Differentiating, we get

$$\frac{dS}{dw} = K \frac{(w^2 + 225)^{3/2} - 3w^2(w^2 + 225)^{1/2}}{(w^2 + 225)^3}.$$

The derivative is defined for all $w$ and is only equal to zero when the numerator is zero. Factoring the common factor of $(w^2 + 225)^{1/2}$ gives

$$(w^2 + 225)^{1/2}(w^2 + 225 - 3w^2),$$

and since $(w^2 + 225)$ is never zero, we must have

$$2w^2 = 225,$$

or

$$w = \pm \sqrt{\frac{225}{2}} = \pm \frac{15}{\sqrt{2}}.$$

Since $w$ is a length, we discard the negative root, and now must test the one critical point $w = \frac{15}{\sqrt{2}}$. Note that for $w < \frac{15}{\sqrt{2}}$, the first derivative is positive, and for $w > \frac{15}{\sqrt{2}}$ the derivative is negative. Thus, by the first derivative test, $w = \frac{15}{\sqrt{2}}$ is a local maximum. Since the function is continuous and this is the only critical point on the domain, $w = \frac{15}{\sqrt{2}}$ is the global maximum. The mic should be placed $\frac{15}{\sqrt{2}} \approx 10.6$ feet from the wall to maximize the sensitivity to sounds at $X$. 
1. [7 points] Liam wants to build a rectangular swimming pool behind his new house. The pool will have an area of 1600 square feet. He will have 8-foot wide decks on two sides of the pool and 10-foot wide decks on the other two sides of the pool (see the diagram below).

![Diagram of a pool with decks]

**Solution:**

a. [4 points] Let $\ell$ and $w$ be the length and width (in feet) of the pool area including the decks as shown in the diagram. Write a formula for $\ell$ in terms of $w$.

$$\ell = \frac{1600}{w - 16} + 20$$

b. [3 points] Write a formula for the function $A(w)$ which gives the total area (in square feet) of the pool and the decks in terms of only the width $w$. Your formula should not include the variable $\ell$. (This is the function Liam would minimize in order to find the minimum area that his pool and deck will take up in his yard. You do not need to do the optimization in this case.)

$$A(w) = \left( \frac{1600}{w - 16} + 20 \right) w$$
7. (14 points) No matter what is done with the other exhibits, the octopus tank at the zoo must be rebuilt. (The current tank has safety issues, and there are fears that the giant octopus might escape!) The new tank will be 10 feet high and box-shaped. It will have a front made out of glass. The back, floor, and two sides will be made out of concrete, and there will be no top. The tank must contain at least 1000 cubic feet of water. If concrete walls cost $2 per square foot and glass costs $10 per square foot, use calculus to find the dimensions and cost of the least-expensive new tank. [Be sure to show all work.]

GIANT OCTOPUS (Enteroctopus)\(^2\)

Denote the width of the tank by \(x\) and the length of the tank by \(y\). The height of the tank is given as 10 feet. We know that the volume of the tank must be at least 1000 cubic feet, so let \(V\) denote the desired volume of the tank, where \(V \geq 1000\). Then for a fixed value of \(V\), we know that \(10xy = V\), so that \(x\) and \(y\) are related by the equation \(x = \frac{V}{10y}\). Now assuming that one of the \(y \times 10\) sides is the front of the tank (i.e., the glass panel), the total cost of the tank is given by:

\[
C = 10(10y) + 2[(2)10x + 10y + xy] = 120y + 40x + 2xy.
\]

Substituting for \(x\), we can write \(C\) as a function of one variable:

\[
C(y) = 120y + \frac{4V}{y} + \frac{V}{5}
\]

Since the cost function increases as \(V\) increases, in order to minimize the cost to build the tank we must have \(V\) be as small as possible, so we set \(V = 1000\). Our cost equation is now:

\[
C = 120y + \frac{4000}{y} + 200
\]

Taking the derivative and setting it equal to zero,

\[
\frac{dC}{dy} = 120 - \frac{4000}{y^2} = 0,
\]

we find that the cost function has a positive critical point at \(y = \sqrt{\frac{100}{3}} \approx 5.774\) feet. Since the second derivative of the cost function, \(\frac{d^2C}{dy^2} = \frac{24000}{y^3}\), is positive for positive values of \(y\), we know that the function is concave up for all \(y > 0\) and this value of \(y\) is a minimum for the cost function. Solving for \(x\), we find \(x \approx 17.321\) feet.

Thus, the glass side is the small side, and the dimensions and cost are:

Dimensions: \(5.774 \times 17.321 \times 10\) feet

Minimum Cost: \(~1585.74\) dollars

\(^2\)See http://www.cephbase.utmb.edu/Tcp/pdf/anderson-wood.pdf. (They really DO escape....)
3. [12 points] The graph of a portion of \( y = f'(x) \), the derivative of \( f(x) \) is shown below. Note that there is a sharp corner at \( x = B \) and that \( x = H \) is a vertical asymptote. The function \( f(x) \) is continuous with domain \((-\infty, \infty)\).

For each of the questions below, circle all of the available correct answers. (Circle none if none of the available choices are correct.)

a. [2 points] At which of the following six values of \( x \) is the function \( f(x) \) not differentiable?

\[ B \quad C \quad E \quad F \quad [H] \quad I \quad \text{NONE} \]

b. [2 points] At which of the following six values of \( x \) does the function \( f'(x) \) appear to be not differentiable?

\[ A \quad [B] \quad C \quad D \quad E \quad F \quad \text{NONE} \]

c. [2 points] At which of the following nine values of \( x \) does \( f(x) \) have a critical point?

\[ [A] \quad B \quad [C] \quad D \quad E \quad F \quad G \quad [H] \quad I \quad \text{NONE} \]

d. [2 points] At which of the following nine values of \( x \) does \( f(x) \) have a local minimum?

\[ A \quad B \quad C \quad D \quad E \quad F \quad G \quad [H] \quad I \quad \text{NONE} \]

e. [2 points] At which of the following nine values of \( x \) is \( f''(x) = 0 \)?

\[ A \quad B \quad [C] \quad D \quad E \quad F \quad G \quad H \quad I \quad \text{NONE} \]

f. [2 points] At which of the following nine values of \( x \) does \( f(x) \) have an inflection point?

\[ A \quad [B] \quad [C] \quad D \quad E \quad F \quad G \quad H \quad I \quad \text{NONE} \]
7. [12 points] On the axes below are graphed $f$, $f'$, and $f''$. Determine which is which, and justify your response with a brief explanation.

**Solution:** Looking to the far right of the graph, curve I has a critical point where it has a slope of zero. At this x-coordinate neither of the other graphs has a root. This means the derivative of I is not in this figure, so I must be $f''$. Looking to the far left of the graph, II has a local maximum where its derivative is zero. Although III has a root near the same x-value, III changes sign from negative to positive at this point. By the first derivative test, III cannot be the derivative of II. Thus, by process of elimination, II must be $f'$ and III must be $f$.

\[
\begin{align*}
f & : \text{III} \\
f' & : \text{II} \\
f'' & : \text{I}
\end{align*}
\]
2. [9 points] Consider a right triangle with legs of length $x$ ft and $y$ ft and hypotenuse of length $z$ ft, as in the following picture:

![Diagram of a right triangle]

a. [2 points] Suppose that the perimeter of the triangle is 8 ft. Let $A(x)$ give the area of the triangle, in ft$^2$, as a function of the side length $x$. In the context of this problem, what is the domain of $A(x)$? Note that you do not need to find a formula for $A(x)$.

**Solution:** Notice that we can let $x$ be arbitrarily close to 0 and still have a perimeter of 8 ft by making $y$ and $z$ both very close to 4. However, since $z$ is always at least as big as $x$ and since $y$ is positive, $x$ cannot be larger than 4 or or else $x + y + z$ would be greater than 8 ft.

**Answer:** $(0, 4)$

b. [7 points] Suppose instead that the perimeter of the triangle is allowed to vary, but the area of the triangle is fixed at 3 ft$^2$. Let $P(x)$ give the perimeter of the triangle, in ft, as a function of the side length $x$.

(i) In the context of this problem, what is the domain of $P(x)$?

**Solution:** $x$ must be positive, but there is no upper bound on $x$. Even if $x$ is very large, with a small enough $y$, it is still possible for the triangle to have area 3 ft$^2$.

**Answer:** $(0, \infty)$

(ii) Find a formula for $P(x)$. The variables $y$ and $z$ should not appear in your answer. (This is the equation one would use to find the value(s) of $x$ minimizing the perimeter. You should not do the optimization in this case.)

**Solution:** The perimeter of the triangle is $x + y + z$ ft. Since we want it to be a function of $x$ only, we need to use other information to eliminate the other variables.

The area is 3 ft$^2$, so we have $\frac{1}{2}xy = 3$. Solving for $y$ yields $y = \frac{6}{x}$.

Since this is a right triangle, by the Pythagorean Theorem, we have $x^2 + y^2 = z^2$. Solving for $z$ yields $z = \sqrt{x^2 + y^2}$. Using $y = \frac{6}{x}$ allows us to write $z$ in terms of $x$ as $z = \sqrt{x^2 + \left(\frac{6}{x}\right)^2}$.

Finally, then, we have

$$P(x) = x + y + z = x + \frac{6}{x} + \sqrt{x^2 + \left(\frac{6}{x}\right)^2}.$$  

**Answer:** $P(x) =$