1.) Consider the infinite range Ising model $H = -H \sum_{i=1}^{N} \sigma_i - (J/2N) \sum_{i,j=1}^{N} \sigma_i \sigma_j$

Here every spin has equal interaction with every other, and we put J/N to keep the total interaction well behaved. You will show that mean field theory is exact for this model, as you might guess anyway. Use the following trick: prove from the simple integral

$$\exp\left\{\frac{ax^2}{2N}\right\} = \int_{-\infty}^{\infty} dy \exp\left\{-\frac{Nay^2}{2} + axy\right\}$$

that $Z = \int_{-\infty}^{\infty} dy \left[ \frac{2\pi}{N} \beta J \right]^{-1/2} e^{-N\beta L}$ where

$L = Jy^2/2 - k_B T \ln\{2\cosh(\beta[H+Jy])\}$. Now, in the limit $N \to \infty$ use the method of steepest descents to express $Z$ in terms of the extremal values of $L$. Show that $M/N$ is given by the $y$ that minimizes $L$, and relate this to the Weiss form of mean-field theory. (See Huang, Statistical Mechanics).

2.) Consider a Landau free energy with a cubic term - this is allowed by symmetry in some cases, e.g., in nematic liquid crystals:

$$f = a_2 \phi^2 + a_3 \phi^3 + a_4 \phi^4 - h\phi.$$ 

Suppose, as usual, that $a_2$ changes sign at $T_c$. Show that there is a first-order transition at some $T$ above $T_c$, and find the discontinuity of $\langle \phi \rangle$ at that point.

3.) We will shortly define 6 critical exponents. In this problem you will calculate 4 of them in the Landau theory for the ferromagnetic Ising model. (Several of them have been done in class already). a.) Show that the specific heat, $C_v$ has a finite discontinuity at $T_c$. In the language of phase transitions, we say that $C_v \sim T_c - T|^\alpha$, and a finite discontinuity corresponds to $\alpha = 0$. b.) Show that $\langle \phi \rangle \sim |T_c - T|^\beta$ as $T$ goes to $T_c$ from below. Find $\beta$. c.) Show that $\chi \sim |T_c - T|^\gamma$, as $T$ goes to $T_c$ from below. Find $\gamma$. d.) Show that $\gamma = \gamma'$. e.) At $T = T_c$ show that $h \sim <\phi>^\delta$. Find $\delta$.

4.) Classical spin waves: In class I referred to spin waves (or magnons). Here you will work out the classical theory. a.) Use the Heisenberg model in one dimension with nearest neighbor coupling: $H = -J \sum_i S(i) \cdot S(i+1)$. The magnetic moment at site $i$ is $\mu = \mu_i n_i S(i)$. Find a term in $\mathcal{H}$ which looks like the classical expression $-\mu \cdot B_{\text{eff}}$. Write down the effective $B$ field. b.) Recall the classical equation of motion for an angular momentum (in this case, $L = h S(i)$): $dL/dt = \mu \times B_{\text{eff}}$. From this you can write coupled non-linear equations for $dS_x/dt$, $dS_y/dt$, and $dS_z/dt$. c.) Assume $S_z = S >> S_x, S_y$. So we will have $dS_z/dt = 0$, and linear equations for $S_{x,y}$. d.) Look for wave-like solutions. $S_{x,y} = u_{x,y} \exp[i(kx-yt)]$. Find the dispersion relation, and show that $S_x$ and $S_y$ oscillate out of phase (i.e. $u_y = iu_x$. This corresponds to precession around the z-axis). e.) Find the dispersion relation for a simple cubic lattice.