NOTE: This handout is intended to help students who are interested in understanding the derivation of the long-run exchange rate determination model presented in class. If you find this material difficult, do not despair, it is not necessary to understand the mathematics behind the model in order to be able to use the model to understand exchange rate behavior.

We begin with the domestic money demand equation. Recall that initially we simply stated that money demand ($M^d$) is a function of the price index ($P$), income ($Y$) and the interest rate ($i$). We did not specify a functional form, but simply stated that $M^d = P \cdot L(Y, i)$. In order to include the money demand function in our model, we need to specify a functional form. The most commonly used functional form (based on empirical evidence) is, $M^d = P^\alpha \cdot e^{-\beta i}$ where $e=2.718281828$ (termed Euler's number), and $\alpha$ and $\beta$ are parameters. This particular functional form can then be simplified by taking its natural logarithm, thereby linearizing the expression.

Recall, some of the basic rules of natural logs (ln):

- $\ln(xy) = \ln(x) + \ln(y)$;
- $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$;
- $\ln(x^a) = a \ln(x)$;
- $\ln(e^x) = x$

Our log-linearized domestic money demand equation (now including time subscripts) is:

$$m_t = p_t + \alpha y_t - \beta i_{tk}$$

And, for simplicity, we assume that the foreign money demand equation is identical:

$$m^*_t = p^*_t + \alpha y^*_t - \beta^*_i i_{tk}$$

where lower case letters denote the natural logarithm (e.g. $\ln(M)=m$), with the exception of the k-period-ahead interest rate ($i_{tk}$). Foreign variables are denoted with an asterisk *.

In these equations, the parameter $\alpha$ can be interpreted as the elasticity of money demand with respect to income (or the percentage change in money demand caused by a 1% change in income). And, the parameter $\beta$ can be interpreted as the responsiveness of money demand to a change in the interest rate.

The third equation to be included in our model is *Purchasing Power Parity* (PPP); which specifies the domestic currency per foreign currency spot exchange rate ($S$) to be equal to the domestic price index ($P$) divided by the foreign price index ($P^*$): $S = P/P^*$.

The log-linearized version of PPP (including time subscripts) is:

$$s_t = p_t - p^*_t$$
We can now solve the domestic and foreign money demand equations for prices and substitute these in the PPP equation to yield the exchange rate as a function of relative money supplies, income and interest rates.

\[ s_t = (m_t^* - m_t^i) + \alpha(y_t^* - y_t^i) + \beta(s_{t+k}^f - s_t) \]  

(4)

Next we include the Uncovered Interest Parity (UIP) condition:

\[ i_{t+k}^* - i_{t+k} = s_{t+k}^f - s_t \]  

(5)

where \( s_{t+k}^f \) is the natural log of the expected k-period-ahead spot exchange rate. (To understand why the expected percentage change in the spot exchange rate is equivalent to the logarithmic difference, see page 4 of this handout.)

If we substitute UIP in equation (4), we arrive at an expression for the exchange rate in terms of relative monies, income and the expected future exchange rate change.

\[ s_t = (m_t^* - m_t^i) + \alpha(y_t^* - y_t^i) + \beta(s_{t+k}^f - s_t) \]  

(6)

We now need to rearrange terms and solve for the reduced form of the exchange rate. If we solve equation (6) for the current spot rate, we have:

\[ s_t = \left( \frac{1}{1 + \beta} \right) \left[ (m_t^* - m_t^i) + \alpha(y_t^* - y_t^i) + \beta s_{t+k}^f \right] \]  

(7)

This equation shows that a change in expectations can cause today's exchange rate to change even in the absence of any change in today's fundamentals. (Money differentials and income differentials are termed "the fundamentals"). But what influences these expectations? In order to answer this question, we need to go a bit further. Begin by simplifying expression (7) by substituting the one-period ahead expectation of the spot exchange rate \( s_{t+k}^f \) for the k-period ahead expectation. Next, move this expression one period into the future and take the expectation. We find that the expected exchange rate in period \( t+1 \) is:

\[ s_{t+1}^f = \left( \frac{1}{1 + \beta} \right) \left[ (m_t^* - m_t^i) + \alpha(y_t^* - y_t^i) + \beta s_{t+2}^f \right] \]

And if we substitute this expression back in (7) and repeat the process iteratively, we get the following infinite series:

\[ s_t = \left( \frac{1}{1 + \beta} \right) \left[ z_t + \left( \frac{\beta}{1 + \beta} \right) z_{t+1}^f + \left( \frac{\beta}{1 + \beta} \right)^2 z_{t+2}^f + \ldots \right] \]

where \( z_t = (m_t^* - m_t^i) + \alpha(y_t^* - y_t^i) \).
This expression, in turn, can be written more compactly using the term $\Sigma$ to denote summation.

$$\sum_{k=0}^{\infty} \frac{\beta}{1+\beta} \left[ \frac{\beta}{1+\beta} \right]^k \left[ (m_{t+k}^e - m_{t+k}^*) \alpha (y_{t+k}^e - y_{t+k}^*) \right]$$

(8)

So, we find that exchange rate expectations are determined by expectations of the fundamentals. And, more generally, the current spot rate depends on expectations of these fundamentals from now into the indefinite future.

Now, if we want to take output market considerations into account, we need to include our definition of the real exchange rate in the model. Recall that the real exchange rate ($Q$) is equal to the nominal exchange rate ($S$) times the representative foreign price index ($P^*$) over the representative domestic price index ($P$). In log-linearized form, the real exchange rate ($q$) is:

$$q_i = s_i + p_i^* - p_i$$

(9)

where $p$ and $p^*$ now represent natural log indices of the prices of the typical domestic and foreign consumption baskets.

We can rearrange this expression in terms of the log of the nominal exchange rate:

$$S_i = q_i + p_i^* - p_i$$

Now we can combine this (revamped PPP equation) with the log-linear money demand equations and UIP to get the final version of the long-run exchange rate model:

$$S_i = \frac{1}{1+\beta} \sum_{k=0}^{\infty} \frac{\beta}{1+\beta} \left[ q_{t+k}^e + (m_{t+k}^e - m_{t+k}^*) + \alpha (y_{t+k}^e - y_{t+k}^*) \right]$$

(10)

In this final form of our exchange rate determination model, the natural log of the expected future real exchange rate appears as an additional factor that influences the natural log of the nominal exchange rate. In this way we are able to include non-monetary influences in the model.

In order to derive the Augmented Fisher Equation, we express the real exchange rate equation (expression (9) above) in terms of expected changes from period t to t+k:

$$[q_{t+k}^e - q_i] = [s_{t+k}^e - s_i] + [p_{t+k}^* - p_i^*] - [p_{t+k}^e - p_i^e]$$

Now, substitute our notation for inflation, $\pi^e$ and $\pi^e*$, for the price change terms:

$$[q_{t+k}^e - q_i] = [s_{t+k}^e - s_i] + [\pi_{t+k}^e - \pi_{t+k}^e*]$$
And, substitute UIP for the expected exchange rate change to arrive at the Augmented Fisher Equation:

\[ [i_{t+k}^e - i_{t+k}^s] = [q_{t+k}^e - q_t] + [\pi_{t+k}^e - \pi_{t+k}^s] \]

*The Relationship Between Percentage Change and Log Differences*

Under what conditions are log differences valid approximations to percentage changes? Or, specifically, when will the following expression hold:

\[
\frac{S_{t+k}^e - S_t}{S_t} \approx \ln (S_{t+k}^e) - \ln (S_t)
\]

where \( \approx \) denotes approximately equal.

Begin by considering the function \( f(x) = \ln(1+x) \). Evaluate a first-order Taylor approximation to this function in the region in which \( x \) is close to zero. (A first order Taylor approximation to \( f(x) \) in the neighborhood of \( x \approx 0 \) is: \( f(0) + f'(x)x \).) The first derivative of the function is:

\[ f'(x) = \frac{1}{1+x} \]

so the first-order Taylor approximation is:

\[ f(x) = \ln (1 + x) \approx \ln (1) + \frac{x}{1 + x} \]

Since \( \ln(1) = 0 \), the value of the function when \( x \) is close to zero is:

\[ \lim_{x \to 0} \ln (1 + x) = x. \]

Now consider the ratio of the expected future spot rate \( S_{t+k}^e \) and the spot rate \( S_t \):

\[ \frac{S_{t+k}^e}{S_t} = 1 + \left( \frac{S_{t+k}^e - S_t}{S_t} \right) \]

As long as the expression in parentheses is small, reflecting a small expected change in the spot exchange rate, we can replace \( x \) by this expression in our formula above, so that:

\[ \ln \left( \frac{S_{t+k}^e}{S_t} \right) \approx \frac{S_{t+k}^e - S_t}{S_t} \]

And, since \( \ln(S_{t+k}^e / S_t) = \ln(S_{t+k}^e) - \ln(S_t) \), the difference of logarithms is a valid approximation to small percentage changes.