1. The Hamiltonian for a vibrating string can be expressed

\[ H = \sum_{n=1}^{\infty} \left( \frac{1}{2}p_n^2 + \frac{1}{2}\omega_n^2q_n^2 \right), \]

where \( \omega_n \) is the frequency of the \( n \)th mode and the operators \( p_n \) and \( q_n \) satisfy the usual commutation relations

\[ [q_n, q_m] = [p_n, p_m] = 0 \quad \text{and} \quad [q_n, p_m] = i\delta_{nm}. \]

Let us define raising and lowering operators \( a_n \) and \( a_n^\dagger \) by

\[ a_n = \frac{i}{\sqrt{2}} \left( \frac{1}{\omega_n}p_n - i\sqrt{\omega_n}q_n \right) \quad \text{and} \quad a_n^\dagger = -\frac{i}{\sqrt{2}} \left( \frac{1}{\omega_n}p_n + i\sqrt{\omega_n}q_n \right). \]

We are to check the commutation relations of these operators, express the Hamiltonian in terms of \( \omega_n \) where \( \omega_n = \sqrt{\omega_n^2} \).

The Hamiltonian for a vibrating string can be expressed

\[ H = C_1 p_n p_m + C_2 q_n q_m + C_3 p_n q_m + C_4 q_n p_m, \]

where \( C_i \) are arbitrary (commuting) functions of the indices \( m \) and \( n \). This expression is clearly symmetric under the interchange of \( m, n \), so that we have

\[ : [a_n, a_m] = 0. \]

An identical argument—replacing \( a_n \) with \( a_n^\dagger \)—shows that

\[ : [a_n, a_n^\dagger] = 0. \]

This argument does not hold, however, for \([a_n, a_n^\dagger] \) because \( a_n a_n^\dagger \) is not merely a permutation of the indices on \( a_n a_n^\dagger \). Nevertheless, looking at the definition of \( a_n, a_n^\dagger \), we see that \( a_n^\dagger a_n = \overline{a_n a_n^\dagger} \). Therefore, the commutator is just the 2i times the imaginary part of \( a_n a_n^\dagger \), which is just the coefficient of \( q_n p_m \) in the expansion of \( a_n a_n^\dagger \). That is,

\[ [a_n, a_n^\dagger] = 2i \left( -\frac{i}{2} \sqrt{\omega_n^2} \delta_{mn} \right), \]

so that

\[ : [a_n, a_n^\dagger] = \delta_{mn}. \]

Now, notice that

\[ a_n a_n^\dagger + a_n^\dagger a_n = \frac{1}{\omega_n} p_n^2 + \omega_n q_n^2, \]

so that

\[ : H = \sum_{n=1}^{\infty} \frac{1}{2\omega_n} \left( a_n a_n^\dagger + a_n^\dagger a_n \right). \]

\(^1\)We are making use of the fact that, as we have defined the raising and lowering operators, \( \overline{a_n} = a_n^\dagger \), if \( p_n \) and \( q_n \) are Hermitian, which they are.
If we posit that a ground state $|0\rangle$ exists such that $a_n|0\rangle = 0$ then we see that its energy is divergent:

$$H|0\rangle = \sum_{n=1}^{\infty} \frac{1}{2} \omega_n \left( a_n a_n^\dagger + a_n^\dagger a_n \right) |0\rangle,$$

$$= \sum_{n=1}^{\infty} \frac{1}{2} \omega_n a_n a_n^\dagger |0\rangle,$$

$$= \left( \sum_{n=1}^{\infty} \frac{1}{2} \omega_n \right) |0\rangle + \left( \sum_{n=1}^{\infty} \frac{1}{2} \omega_n a_n a_n^\dagger \right) |0\rangle,$$

$$= \left( \sum_{n=1}^{\infty} \frac{1}{2} \omega_n \right) |0\rangle.$$

This is a divergent ‘vacuum energy.’ A consistent prescription to remove this (infinite) vacuum energy is ‘normal ordering,’ where we require that annihilation operators always act on a state first. This is done using the canonical commutation relations:

$$H = \sum_{n=1}^{\infty} \frac{1}{2} \omega_n \left( a_n a_n^\dagger + a_n^\dagger a_n \right) = \sum_{n=1}^{\infty} \frac{1}{2} \omega_n \left( 2 a_n^\dagger a_n + \delta_{mn} \right).$$

After subtracting the divergent piece, we obtain

$$H \equiv \sum_{n=1}^{\infty} \omega_n a_n a_n^\dagger.$$

Because $a_n^\dagger$ commutes with $a_m$ and $a_m^\dagger$ for all $m \neq n$, we see that

$$H a_n^\dagger = \sum_{m=1}^{\infty} \omega_m a_m a_n^\dagger,$$

$$= \sum_{m=1}^{\infty} \omega_m a_m \left( \delta_{mn} + a_m^\dagger a_m \right),$$

$$= a_n^\dagger \left( \omega_n + H \right).$$

This implies that the (energy) eigenvalue of $a_\ell^\dagger |\xi\rangle$ is $\omega_\ell$ greater than the eigenvalue of $|\xi\rangle$. In particular, this implies that the energy of the state

$$|\ell_1, \ell_2, \ldots, \ell_N\rangle = \left( a_1^\dagger \right)^{\ell_1} \left( a_2^\dagger \right)^{\ell_2} \cdots \left( a_N^\dagger \right)^{\ell_N} |0\rangle,$$

is simply

$$H|\ell_1, \ell_2, \ldots, \ell_N\rangle = \left( \sum_{n=1}^{N} \ell_n \omega_n \right) |\ell_1, \ell_2, \ldots, \ell_N\rangle.$$
2. We are to show that the energy-momentum four-vector operator for a quantized Klein-Gordon field $\varphi$ can be put into the form

$$P^\mu = \int \frac{d^3p}{(2\pi)^3} p^\mu a_p^\dagger a_p.$$  

We should explain the significance of the creation and annihilation operators and state their commutation relations. Using this, we are to show that if $\varphi$ is the in the Heisenberg picture, then

$$[P^\mu, \varphi(x)] = -i\partial^\mu \varphi(x),$$  

which is consistent with the result

$$\varphi(x) = e^{ipx} \varphi(t_0) e^{-ipx}.$$  

This problem is solved almost explicitly in Peskin and Schroeder so we may be somewhat scant in our work here. First, we remark that by our work in class $H = P^0 = \int \frac{d^3p}{(2\pi)^3} p^0 a_p^\dagger a_p,$  

where $p^0 = E_p$. Furthermore, in general,

$$P^i = \int p^0 d^3x = -\int d^3x \pi \partial_i \phi,$$

where $\pi$ is the canonically conjugate field $\pi(x)$ to $\varphi(x)$. Therefore, by direct computation using the expressions for $\pi(x)$ and $\varphi(x)$ in terms of creation and annihilation operators, we have

$$P^i = -\int \frac{d^3p}{(2\pi)^3} \int d^3q e^{i(p+q)x} p^i \frac{E_q}{E_p} \left( a_q - q^+_\dagger q - a^+_p - a^-_p \right),$$

$$= -\int \frac{d^3p}{(2\pi)^3} \int d^3q \left( a^+_p - a^-_p \right) \left( a^+_p - a^-_p \right),$$

$$= \int \frac{d^3p}{(2\pi)^3} \int d^3q \left( -a^+_p e^{-ipx} + a^-_p e^{ipx} \right),$$

where in the last line we have used the fact that when considering $\langle 0 | P^i | 0 \rangle$ all the other terms from will vanish. This is what we were required to prove.

We can also evaluate the commutator $[p^\mu, \varphi(x)]$, we may proceed by direct computation.

$$[P^\mu, \varphi(x)] = \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{\sqrt{2E_p}} p^\mu \left( a_q a^+_q (a_p e^{-ipx} + a^+_p e^{ipx}) - (a_p e^{-ipx} + a^+_p e^{ipx}) a^+_q a^+_q \right),$$

$$= \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{\sqrt{2E_p}} p^\mu \left( [a_q, a^+_p] e^{ipx} - a_q [a_p, a^+_q] e^{-ipx} \right),$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} p^\mu \left( -a_p e^{-ipx} + a^+_p e^{ipx} \right).$$

It does not take much effort to show that this is equivalent to $-i\partial^\mu \varphi(x)$. And because our representation of $\varphi$ has been within the framework of the Heisenberg picture, it is obviously consistent with the result $\varphi(x) = e^{ipx} \varphi(t_0) e^{-ipx}$. 
3. The number operator $\mathcal{N}$ in quantized Klein-Gordon theory is defined by
$$\mathcal{N} = \int \frac{d^3p}{(2\pi)^3} a_p^\dagger a_p.$$  
We are to calculate the commutator of $\mathcal{N}$ with $a_q^\dagger$ and $a_q$ and show that this justifies the name.

Using the commutation relations for the creation and annihilation operators $a_p$ and $a_p^\dagger$, we see that,
$$\mathcal{N} a_q^\dagger = \int \frac{d^3p}{(2\pi)^3} a_p^\dagger a_p a_q^\dagger,$$
$$= \int \frac{d^3p}{(2\pi)^3} a_p^\dagger \left( (2\pi)^3 \delta^{(3)}(p-q) + a_q^\dagger a_p \right),$$
$$= a_q^\dagger (1 + \mathcal{N}).$$

A nearly identical calculation shows that
$$\mathcal{N} a_q = a_q (-1 + \mathcal{N}).$$

It is not hard to see that $\mathcal{N}|0\rangle = 0$. Therefore, we see that the eigenvalue of $\mathcal{N}$ is simply the total number of excitations in a given state.

4. Consider a triplet of scalar fields $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ with the Lagrangian$^2$
$$\mathcal{L} = \frac{1}{2} (\partial_\mu \varphi)^2 - \frac{1}{2} m^2 \varphi^2.$$  
We are to compute the field $\pi$ canonically conjugate to $\varphi$ and obtain the Hamiltonian—this should be expressible in terms of canonical ladder operators. Using this, we should show that each of the component fields separately obey the Klein-Gordon equation.

Finally, we are to calculate the Noether currents and conserved charges associated with the $SO(3)$ invariance of the theory.

First, we see directly that we have a triplet of fields $\pi(x) = (p_1, p_2, p_3)$ where
$$\pi_i(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_i} = \partial_\mu \varphi_i.$$

In our notation it is abundantly obvious that the Hamiltonian is exactly what we expect from $\varphi^4$-theory which we have worked out in class, namely,
$$H = \int d^3x \left( \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \varphi)^2 + \frac{1}{2} m^2 \varphi^2 \right) = \int d^3p \frac{(2\pi)^3}{(2\pi)^3} E_p a_p^\dagger a_p,$$
where in the last expression we have implemented normal ordering and $a_p, a_p^\dagger$ are each triplets of operators and the summation over the index $i$ is implied. Again by analogy to $\varphi^4$-theory, we see that each field separately obeys the Klein-Gordon equation.

From our work in example sheet 1, we know that the conserved current is simply
$$j_i^\mu = -\epsilon_{ijk} \varphi_j \partial^\mu \varphi_k.$$

Thus the conserved current can be computed directly to be
$$Q_i = -\epsilon_{ijk} \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \frac{iE_q}{2 \sqrt{E_p E_q}} e^{\pm i(p+q)} \left( -a_p a_{qk} + a_{p}^\dagger a_{q}^\dagger \right) e^{\pm i(p-q)} \left( a_p a_{qk}^\dagger - a_{p}^\dagger a_{qk} \right),$$
$$= -\epsilon_{ijk} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \left( a_p a_{p-k} = a_{p}^\dagger a_{p-k}^\dagger + a_{p}^\dagger a_{p-k} - a_{p}^\dagger a_{p-k} \right),$$
$$= i\epsilon_{ijk} \int \frac{d^3p}{(2\pi)^3} a_{p}^\dagger a_{p-k},$$
where in the last line we have used normal ordering. The commutation relations between the $Q_i$ with $\varphi_j$ and each other follow readily from calculations similar to those done in exercise 2 above.

$^2$We use the obvious Euclidean metric on $\varphi$. Notice that this notation will conceal—and make much more natural—the results of this exercise.
5. If the one-particle states of a Klein-Gordon field theory are given by

\[ |p\rangle = \sqrt{2E_p} a_p^\dagger |0\rangle, \]

then

\[ \langle 0| \varphi(x)|p\rangle = e^{ipx} \quad \text{and} \quad \langle p|q\rangle = 2E_p^2 (2\pi)^3 \delta(3)(p-q), \]

which implies that

\[ \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |p\rangle \langle p| = \text{id}, \]

or, a complete set of states. We are to explain what this has to do with the Lorentz invariance of the normalization of $|p\rangle$.

Let us begin by direct calculation. We will use frequently the fact that $\langle 0| a^\dagger_p = 0$—which is the conjugate condition to the ground state defined by $a_p |0\rangle = 0$.

\begin{align*}
\langle 0| \varphi(x)|p\rangle &= \langle 0| \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} e^{iqx} \left( a_q + a_q^\dagger \right) \sqrt{2E_p} a_p^\dagger |0\rangle, \\
&= \langle 0| \int \frac{d^3q}{(2\pi)^3} \sqrt{\frac{E_p}{E_q}} e^{iqx} \left( (2\pi)^3 \delta(3)(p-q) + a_q^\dagger a_p + a_p a_q^\dagger \right) |0\rangle, \\
&\therefore \langle 0| \varphi(x)|p\rangle = e^{ipx}.
\end{align*}

By a similar calculation, we see that

\begin{align*}
\langle p|q\rangle &= 2\sqrt{E_p E_q} \langle 0| a_p a_q^\dagger |0\rangle, \\
&= 2\sqrt{E_p E_q} |0\rangle \left( (2\pi)^3 \delta(3)(p-q) + a_p a_q^\dagger \right) |0\rangle, \\
&\therefore \langle p|q\rangle = 2E_p (2\pi)^3 \delta(3)(p-q).
\end{align*}

It is obvious that this relation implies that

\[ \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |p\rangle \langle p|q\rangle = |q\rangle. \]

This is the statement that any state $|q\rangle$ can be decomposed into a complete set of states $|p\rangle \langle p|$. Notice that the decomposition is made using the Lorentz-invariant measure $\frac{d^3p}{2E_p}$; this implies that the components of any state $|q\rangle$ with respect to the complete set $|p\rangle \langle p|$ are Lorentz-invariant.
6. The retarded Klein-Gordon propagator is given by
\[ D_R(x - y) = \theta(x^0 - y^0)|\langle \varphi(x), \varphi(y) | 0 \rangle|, \]
where \( \varphi(x) \) and \( \varphi(y) \) are time-dependent fields. We are to show that
\[ D_R(x - y) = \theta(x^0 - y^0) \int \frac{d^4p}{(2\pi)^4} \frac{1}{2E_p} \left( e^{-ip(x-y)} - e^{ip(x-y)} \right), \]
where \( p^0 = E_p \) in the exponents, can be reexpressed as a four-momentum integral
\[ D_R(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip(x-y)}, \]
with a particular prescription for the contour.

Let us first compute the commutator \( [\varphi(x), \varphi(y)] \). Notice that the creation and annihilation operators commute with themselves respectively and so only the mixing terms can contribute. Therefore, we see that
\[
[\varphi(x), \varphi(y)] = \int \frac{d^3p \, d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_q}} \left( e^{-ipx}e^{-iE_p}a_p^\dagger a_q^\dagger + e^{ipx}e^{-iE_p}a_p a_q \right),
\]
\[
= \int \frac{d^3p \, d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_q}} \left( e^{-ipx+iqy}(2\pi)^3 \delta(3)(p - q) - e^{ipx-iqy}(2\pi)^3 \delta(3)(p - q) \right),
\]
\[
= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left( e^{-ip(x-y)} - e^{-ip(x-y)} \right).
\]

Noting that this operator acts trivially on the state \( |0\rangle \) and that \( \langle 0 | = 1 \), we obtain the desired relation.

Consider now the four-momentum integral:
\[
\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip(x-y)},
\]
The integral over \( p^0 \) has poles at \( p^0 = \pm E_p = \sqrt{p^2 + m^2} \). The residues of the poles are easily determined:
\[
\text{Res}(p^0 = +E_p) = \frac{i}{2E_p} e^{-iE_p(x^0 - y^0) + ip(\vec{x} - \vec{y})} \quad \text{and} \quad \text{Res}(p^0 = -E_p) = -\frac{i}{2E_p} e^{iE_p(x^0 - y^0) + ip(\vec{x} - \vec{y})}.
\]

To evaluate this integral, we must give a prescription to avoid the poles—and this prescription is not canonical. Let us take for the sake of argument the contour which passes above both poles in the complex plane. This prescription is made explicit by setting
\[
\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip(x-y)} \mapsto \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon p^0} e^{-ip(x-y)},
\]
where \( \epsilon \) is a small, real number\(^3\). The integral is essentially the same as that which gives the Feynman propagator—only the contour prescription is different.

Notice that if \( x^0 - y^0 < 0 \), then the contour cannot be closed downward—the exponential will be unbounded—and so the only option is to close the contour in the \( \text{Im}(p^0) > 0 \) half-plane. This contour encloses neither pole and so by Cauchy’s theorem, the integral vanishes. Alternatively, if \( x^0 - y^0 > 0 \) then the contour must be closed downward—by the same argument—and hence both poles will be enclosed.

Therefore, by Cauchy’s theorem, the integral gives
\[
\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon p^0} e^{-ip(x-y)} = \theta(x^0 - y^0) \int \frac{d^4p}{(2\pi)^3} \frac{1}{2E_p} \left( e^{iE_p(x^0 - y^0) + ip(\vec{x} - \vec{y})} - e^{-iE_p(x^0 - y^0) + ip(\vec{x} - \vec{y})} \right).
\]

Using spherical symmetry, we may change variables in the first term of the integrand and obtain
\[
\therefore D_R(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon p^0} e^{-ip(x-y)} = \theta(x^0 - y^0) \int \frac{d^4p}{(2\pi)^3} \frac{1}{2E_p} \left( e^{-ip(x-y)} - e^{ip(x-y)} \right).
\]
\(^3\)This prescription is well-defined if the field is massive.
7. We are to show that the matrices $S^{\kappa\lambda} = \frac{1}{4} [\gamma^\kappa, \gamma^\lambda]$ satisfy the Lorentz algebra.

Let us first use recall two elementary identities of the Lie bracket (both are trivial consequences of the Jacobi identities):


These allow us to derive a very useful identity involving matrices satisfying the Clifford-Dirac algebra:

\[ \left[ \gamma^\kappa \gamma^\lambda, \gamma^\mu \gamma^\nu \right] = \left[ \gamma^\kappa, \gamma^\mu \right] \gamma^\nu + \gamma^\mu \left[ \gamma^\kappa, \gamma^\lambda, \gamma^\nu \right], \]

\[ = \gamma^\kappa \left\{ \gamma^\lambda, \gamma^\mu \right\} \gamma^\nu - \left\{ \gamma^\kappa, \gamma^\mu \right\} \gamma^\lambda \gamma^\nu + \gamma^\mu \gamma^\kappa \left\{ \gamma^\lambda, \gamma^\nu \right\} - \gamma^\mu \left\{ \gamma^\kappa, \gamma^\nu \right\} \gamma^\lambda, \]

\[ = 2g^{\kappa\mu} \gamma^\kappa \gamma^\nu - 2g^{\kappa\mu} \gamma^\lambda \gamma^\nu + 2g^{\kappa\nu} \gamma^\kappa \gamma^\lambda - 2g^{\kappa\nu} \gamma^\mu \gamma^\lambda. \]

Now, let us evaluate $[S^{\kappa\lambda}, S^{\mu\nu}]$. Notice that we can write $S^{\kappa\lambda} = \frac{1}{4} (\gamma^\kappa \gamma^\lambda - g^{\kappa\lambda})$ and that $g^{\kappa\lambda}$ obviously commutes with $g^{\mu\nu}$ and with $\gamma^\mu \gamma^\nu$ so that

\[ [S^{\kappa\lambda}, S^{\mu\nu}] = \frac{1}{4} \left[ \gamma^\kappa \gamma^\lambda, \gamma^\mu \gamma^\nu \right] = \frac{1}{2} \left( g^{\lambda\mu} \gamma^\kappa \gamma^\nu - g^{\kappa\mu} \gamma^\lambda \gamma^\nu + g^{\lambda\nu} \gamma^\mu \gamma^\kappa - g^{\kappa\nu} \gamma^\mu \gamma^\lambda \right). \]

If we make the substitution $\gamma^\kappa \gamma^\lambda = 2S^{\kappa\lambda} + g^{\kappa\lambda}$, & etc in the expression above, we see that

\[ [S^{\kappa\lambda}, S^{\mu\nu}] = \frac{1}{2} \left( g^{\lambda\mu} S^{\kappa\nu} \gamma^\nu - g^{\kappa\mu} S^{\lambda\nu} \gamma^\nu + g^{\lambda\nu} S^{\mu\kappa} \gamma^\kappa - g^{\kappa\nu} S^{\mu\lambda} \right), \]

\[ = \gamma^\lambda S^{\kappa\nu} \gamma^\nu + 2g^{\lambda\mu} S^{\kappa\nu} \gamma^\nu + 2g^{\kappa\mu} S^{\lambda\nu} \gamma^\nu + 2g^{\lambda\nu} S^{\mu\kappa} \gamma^\kappa + 2g^{\kappa\nu} S^{\mu\lambda} - \frac{1}{2} \frac{1}{2} g^{\kappa\nu} g^{\mu\lambda}. \]

Therefore, $S^{\mu\nu}$ satisfies the Lorentz algebra.

8. Let $S^i \equiv \frac{1}{4} \epsilon_{ijk} \gamma^j \gamma^k$. We are to show that $[S^i, S^j] = i\epsilon_{ijk} S^k$, $[\gamma^0, S^i] = [\gamma^5, S^i] = 0$ and that $(S^i)^2 = \frac{1}{4}$ for each $i$. We should verify these results for a particular representation and discuss spin.

Let us note that as defined we have

\[ S^1 = S^{23} \quad S^2 = S^{31} \quad S^3 = S^{12}. \]

Notice that by the antisymmetry of the commutator bracket, it is sufficient for us to calculate $[S^i, S^{i+1}]$ where $i \in \{1, 2, 3\}$. Using the definition of the bracket and the fact that $S^{\mu\nu}$ satisfy the Lorentz algebra condition (see example sheet 1), we have

\[ [S^i, S^{i+1}] = \left[ S^{(i+1)(i+2)}, S^{i+2} \right] = ig^{(i+2)(i+3)} S^{(i+1)} = -iS^{(i+1)} = iS^{(i+1)} = iS^{(i+3)}. \]

Noticing that this expression is completely antisymmetric we have shown that

\[ \vdots [S^i, S^j] = i\epsilon_{ijk} S^k. \]

The other identities are trivial.
9. We are to verify a handful of ‘useful’ identities of the Dirac algebra\(^4\).

a. \((\gamma^5)^2 = 1\).

Proof:
\[
(\gamma^5)^2 = -\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^4 \gamma^5 \gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^4 \gamma^1 = -\gamma^0 \gamma^1 \gamma^2 \gamma^4 \gamma^1 = 1.
\]

b. \(\phi \psi = 2a \cdot b - \psi \phi = a \cdot b + 2S^{\mu\nu} a_\mu b_\nu\).

Proof:
\[
\phi \psi = a_\mu b_\nu \gamma^\mu \gamma^\nu = a_\mu b_\nu (2g^{\mu\nu} - \gamma^\mu \gamma^\nu) = 2a \cdot b - \psi \phi.
\]

Using \(\gamma^\mu \gamma^\nu = 2S^{\mu\nu} + g^{\mu\nu}\) from above, we see that
\[
\phi \psi = a_\mu b_\nu \gamma^\mu \gamma^\nu = a_\mu b_\nu (2S^{\mu\nu} + g^{\mu\nu}) = a \cdot b + 2S^{\mu\nu} a_\mu b_\nu.
\]

\(\omega\) \(\delta\) \(\tau\) \(\delta\) \(\psi\) \(\delta\) \(\tau\)

c. \(\text{tr}(\phi \psi) = 4a \cdot b\).

Proof:
\[
\text{tr}(\phi \psi) = \text{tr}(2a \cdot b - \psi \phi) = 2a \cdot b - \text{tr}(\psi \phi),
\]

but \(\text{tr}(\psi \phi) = \text{tr}(\phi \psi)\) so
\[
\therefore \text{tr}(\phi \psi) = 4a \cdot b.
\]

\(\omega\) \(\delta\) \(\tau\) \(\delta\) \(\psi\) \(\delta\) \(\tau\)

d. \(\text{tr}(\gamma^5) = 0\).

Proof:
\[
\text{tr}(\gamma^5) = i\text{tr}(\gamma^0 \gamma^1 \gamma^2 \gamma^3) = i\text{tr}(\gamma^3 \gamma^0 \gamma^1 \gamma^2)
\]
by the cyclic properties of the trace, but is equal to
\[
-\text{tr}(\gamma^3 \gamma^0 \gamma^1 \gamma^2)
\]
by commuting \(\gamma^3\) from the left using the anti-commutivity of distinct gamma matrices.
\[
\therefore \text{tr}(\gamma^5) = 0.
\]

\(\omega\) \(\delta\) \(\tau\) \(\delta\) \(\psi\) \(\delta\) \(\tau\)

e. \(\text{tr}(\phi_1 \phi_2 \cdots \phi_r) = 0\) if \(r\) is odd\(^5\).

Proof:
\[
\text{tr}(\gamma^{\mu_1} \gamma^{\mu_2} \cdots \gamma^{\mu_r}) = \text{tr}(\gamma^{\mu_1} \gamma^{\mu_2} \cdots \gamma^{\mu_r} \gamma^5) = \text{tr}(\gamma^5 \gamma^{\mu_1} \cdots \gamma^{\mu_r}),
\]
by the cyclic property of the trace, but by commuting \(\gamma^5\) from the right, we have
\[
\text{tr}(\gamma^5 \gamma^{\mu_1} \cdots \gamma^{\mu_r}) = (-1)^r \text{tr}(\gamma^{\mu_1} \cdots \gamma^{\mu_r} \gamma^5),
\]
which must vanish if \(r\) is odd.

\(\omega\) \(\delta\) \(\tau\) \(\delta\) \(\psi\) \(\delta\) \(\tau\)

f. \(\text{tr}(\phi \psi \phi) = 4[(a \cdot b)(c \cdot d) + (a \cdot d)(b \cdot c) - (a \cdot c)(b \cdot d)].\)

Proof: We first compute the trace of \(\gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}\):
\[
\text{tr}(\gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}) = 2\text{tr}(\gamma^{\mu} \gamma^{\nu} g^{\rho\sigma}) - \text{tr}(\gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}),
\]
\[
= 2\text{tr}(\gamma^{\mu} \gamma^{\nu} g^{\rho\sigma}) - 2\text{tr}(\gamma^{\rho} \gamma^{\mu} g^{\nu\sigma}) + \text{tr}(\gamma^{\rho} \gamma^{\mu} \gamma^{\nu} \gamma^{\sigma}),
\]
\[
= 2\text{tr}(\gamma^{\mu} \gamma^{\nu} g^{\rho\sigma}) - 2\text{tr}(\gamma^{\rho} \gamma^{\mu} g^{\nu\sigma}) + 2\text{tr}(\gamma^{\nu} \gamma^{\rho} g^{\mu\sigma}) - \text{tr}(\gamma^{\rho} \gamma^{\mu} \gamma^{\nu} \gamma^{\sigma}),
\]
\[
\therefore \text{tr}(\gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}) = 4g^{\mu\nu} g^{\rho\sigma} - 4g^{\mu\sigma} g^{\nu\rho} + 4g^{\nu\rho} g^{\mu\sigma}.
\]

From whence it is obvious that
\[
\therefore \text{tr}(\phi \psi \phi) = a_\mu b_\nu c_\rho d_\sigma \text{tr}(\gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}) = 4[(a \cdot b)(c \cdot d) + (a \cdot d)(b \cdot c) - (a \cdot c)(b \cdot d)].
\]

\(\omega\) \(\delta\) \(\tau\) \(\delta\) \(\psi\) \(\delta\) \(\tau\)

\(^4\)Although we were not asked to prove these, the following identities were very useful: \((\gamma^0)^2 = 1\), \((\gamma^r)^2 = -1\) for each \(r\), and \(\prod_i (\gamma^r, \gamma^r) = 0\), and \(\text{tr}(1) = 4\).

\(^5\)Because \(\text{tr}(\phi_1 \cdots \phi_r) = a_{1 \mu_1} \cdots a_{r \mu_r} \text{tr}(\gamma^{\mu_1} \cdots \gamma^{\mu_r})\), it is sufficient to show this holds for an odd number of gamma matrices alone.

\(^6\)Notice that our result from exercise 9.c above implies that \(\text{tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}\).
g. \( \text{tr}(\gamma^5 g \gamma) = 0. \)

**proof:** It is sufficient to show that \( \text{tr}(\gamma^5 \gamma^\mu \gamma^\nu) = 0. \) Using the definition of \( \gamma^5 \) we see that \( \text{tr}(\gamma^5 \gamma^\mu \gamma^\nu) = i \text{tr}(\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^\mu \gamma^\nu). \) If \( \mu, \nu \neq 0 \) then the trace would change sign by commuting \( \gamma^0 \) all the way to the left; but because the trace is cyclic, this cannot change sign so the trace must vanish. If both \( \mu, \nu \neq 0 \) then we can eliminate it from the expression because \( (\gamma^0)^2 = 1 \) but we know that \( \text{tr}(\gamma^5) = 0, \) and so the expression vanishes.

The last case to check is if exactly one of \( \mu \) or \( \nu \) is equal to 0. Up to an overall sign, we may without loss of generality consider the case that \( \nu = 0, \mu \neq 0. \) In this case, \( \text{tr}(\gamma^5 \gamma^\mu \gamma^\nu) = i \text{tr}(\gamma^1 \gamma^2 \gamma^3 \gamma^\mu). \) Using the lemma proved in exercise 9.f above, we see that

\[
\text{tr}(\gamma^1 \gamma^2 \gamma^3 \gamma^\mu) = 4 \left(g^{12} \gamma^\mu - g^{31} \gamma^3 \gamma^\mu + g^{23} \gamma^2 \gamma^1 \gamma^\mu\right) = 0.
\]

This exhausts the possibilities for \( \mu, \nu \) and

\[
\therefore \text{tr}(\gamma^5 \gamma^\mu \gamma^\nu) = 0.
\]

h. \( \overline{\gamma} \gamma^\nu = -2\gamma^\nu. \)

**proof:** \( \overline{\gamma} \gamma^\nu = 2g^{\mu \nu} \gamma_\mu - \gamma^\nu \gamma = 2\gamma^\nu - 4\gamma^\nu = -2\gamma^\nu. \)

i. \( \overline{\gamma} \gamma^\nu \gamma^\rho = 4g^{\rho \rho}. \)

**proof:** \( \overline{\gamma} \gamma^\nu \gamma^\rho = 2g^{\rho \rho} \gamma_\mu \gamma^\nu - \gamma^\nu \gamma^\rho = 2\{\gamma^\rho, \gamma^\nu\} = 4g^{\rho \rho}. \)

j. \( \overline{\gamma} \gamma^\nu \gamma^\rho \gamma = -2\gamma^\sigma \gamma^\rho \gamma. \)

**proof:** \( \overline{\gamma} \gamma^\nu \gamma^\rho \gamma = 2g^{\rho \rho} \gamma_\mu \gamma^\nu - \gamma^\nu \gamma^\rho \gamma^\rho = 4\gamma^\rho g^{\rho \rho} - 2\gamma^\rho \gamma^\rho \gamma^\nu - 4g^{\rho \rho} \gamma^\rho = -2\gamma^\sigma \gamma^\rho \gamma. \)

k. \( \text{tr}(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma) = -4i\epsilon_{\mu \nu \rho \sigma}. \)

**proof:** Notice that if any two of \( \mu, \nu, \rho, \sigma \) were the same, then up to \( \pm 1 \) we could eliminate the pair by commuting them across and exercise 9.g above shows that the trace will vanish. The only way for no two of \( \mu, \nu, \rho, \sigma \) to be the same is if they are a permutation on the numbers \( \{0, 1, 2, 3\} \) and so equal to \( \mp i \gamma^5. \) More specifically, it is clear that \( \gamma^\sigma(g^{(0)} \gamma^0(g^{(1)} \gamma^1(g^{(2)} \gamma^2(g^{(3)} \gamma^3)) = \text{permutation on 4-numbers} \) then \( \gamma^\sigma(g^{(0)} \gamma^0(g^{(1)} \gamma^1(g^{(2)} \gamma^2(g^{(3)} = -\text{sign}(\sigma) i \gamma^5. \)

Therefore, we have shown that \( \text{tr}(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma) \) is totally antisymmetric in the indices \( \mu, \nu, \rho, \sigma \) and for any even permutation of \( \{0, 1, 2, 3\}, \) it is equal to \( -i \text{tr}(1) = -4i \) and for any odd permutation it is \( \text{tr}(1) = 4i. \) Because the totally antisymmetric tensor density \( \epsilon_{\mu \nu \rho \sigma} \) is such symbol, we see that

\[
\therefore \text{tr}(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma) = -4i\epsilon_{\mu \nu \rho \sigma}.
\]