Superficial Divergences

Let us consider $\varphi^3$ scalar field theory in $d = 4$ dimension. The Lagrangian for this theory is

$$L = \frac{1}{2} (\partial_\mu \varphi)^2 - \frac{1}{2} m^2 \varphi^2 - \frac{1}{3!} g \varphi^3.$$ 

**a)** Let us determine the superficial divergence $D$ for this theory in terms of the number of vertices $V$ and the number of external lines $N$. From this we are to show that the theory is super-renormalizable.

In generality, the superficial divergence of a $\varphi^n$ theory in $d$ dimensions can be given by

$$D = dL - 2P,$$

where $L$ is the number of loops and $P$ is the number of propagators because each loop contributes a $d$-dimensional integration and each propagator contributes a power of 2 in the denominator. Furthermore, we see that $nV = N + 2P$ because each external line connects to one vertex and each propagator connects two and each vertex involves $n$ lines. This implies that $P = \frac{1}{2}(nV - N)$.

Therefore, still in complete generality, the superficial divergence of a $\varphi^n$ theory in $d$-dimensions may be written

$$D = dL - 2P = d^2 - nV + 2P = d - nV + N.$$ 

Therefore, in a 4-dimensional $\varphi^3$-theory the superficial divergence is given by

$$D = 4 - V - N.$$ 

We see that because $D \propto -V$ the theory is super-renormalizable.

**b)** We are to show the superficially divergent diagrams for this theory that are associated with the exact two-point function.

Using equation (1.a) above, we see that the three superficially divergent diagrams in this $\varphi^3$-theory associated with the exact two-point function are:

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**c)** Let us compute the mass dimension of the coupling constant $g$.

Because $L$ must have dimension (mass)$^4$ each term should have dimension (mass)$^4$.

Because of the $m^2 \varphi^2$ term, this implies that the field $\varphi$ has dimension (mass)$^1$.

Therefore the coupling $g$ must have dimension (mass)$^1$.

Renormalization and the Yukawa Coupling

We are to consider the theory of elementary fermions that couple to both QED and a Yukawa field $\phi$ governed by the interaction Hamiltonian

$$H_{\text{int}} = \int d^3 x \frac{\lambda}{\sqrt{2}} \bar{\psi} \gamma^\mu \psi + \int d^3 x e A_\mu \bar{\psi} \gamma^\mu \psi.$$ 

**a)** Let us verify that $\delta Z_1 = \delta Z_2$ to the one-loop order.

We computed in homework 4 the amplitude for the $\bar{\psi} \gamma^\mu \psi$ vertex with a virtual scalar $\phi$,

$$i\mathcal{M} = \int \frac{d^4k}{(2\pi)^4} \bar{u}(p') \gamma^\mu \frac{i\lambda}{\sqrt{2}} \frac{i}{(p-k')^2 - m^2 + i\epsilon} \frac{i}{(k'-q)^2 - m^2 + i\epsilon} \frac{i}{(q^2 - m^2 + i\epsilon)} \frac{i}{(p-k)^2 - m^2 + i\epsilon} \frac{i}{(k+q)^2 - m^2 + i\epsilon} \frac{i}{(q+q)^2 - m^2 + i\epsilon} \frac{i\lambda}{\sqrt{2}} u(p).$$
In the limit where \( q \to 0 \), we see that this implies

\[
\pi(p) \delta \Gamma^\mu u(p) = \frac{\lambda^2}{2} \int \frac{d^d k}{(2\pi)^d} \frac{\pi(p) [\gamma^\mu (k^0 + m)] u(p)}{(p - k)^2 - m^2 + ic}(k^2 - m^2 + ic)(k^2 - m^2 + ic) + m^2(1 + z)^2 \].

Using Feynman parametrization to simplify the denominator, we will use the variables

\[
\ell \equiv k - z p \quad \text{and} \quad \Delta \equiv (1 - z)^2 m^2 + zm^2 \phi.
\]

The numerator of the integrand is then reduced to

\[
\mathcal{N} = \pi(p) [\gamma^\mu (k^0 + m)] u(p),
\]

\[
= \pi(p) \left[ \gamma^\mu \gamma^\nu \gamma^\rho + m z \gamma^\mu \gamma^\nu \gamma^\rho + m z \gamma^\mu \gamma^\nu \gamma^\rho \right] u(p),
\]

\[
= \pi(p) \left[ 1 \frac{d^2}{d \ell^2} (2 \gamma^\mu - d \gamma^\nu) + z^2 m^2 \gamma^\mu + m^2 z \gamma^\mu + m^2 z \gamma^\mu + m^2 z \gamma^\mu \right] u(p),
\]

\[
= \pi(p) \left[ \gamma^\mu \left( \frac{2 - d}{d} \ell^2 + m^2 (1 + z) \right) \right] u(p).
\]

Combining this with our work above, we see that this implies

\[
\delta Z_1 = -\delta F_1(q = 0) = -i \frac{\lambda^2}{2} \int \frac{d^d k}{(2\pi)^d} \left[ \frac{(\frac{2 - d}{d}) \ell^2}{\ell^2 - \Delta + i\epsilon} + m^2(1 + z)^2 \right],
\]

\[
= -i \frac{\lambda^2}{2} \int \frac{d^d k}{(2\pi)^d} \left[ 2 - d \frac{d}{d} \frac{i \Gamma(\frac{2 - d}{2})}{\Delta^{-d/2}} - \frac{i m^2(1 + z)^2}{\Delta} \right],
\]

\[
\approx \frac{\lambda^2}{32 \pi^2} \int \frac{d^d k}{(2\pi)^d} \left[ 2 - d \frac{d}{d} \frac{z^2 - \log \Delta - \gamma_E + \log(4\pi)}{\Delta} - \frac{m^2(1 + z)^2}{\Delta} \right],
\]

\[
\therefore \delta Z_1 = \frac{\lambda^2}{32 \pi^2} \int \frac{d^d k}{(2\pi)^d} \left[ \left( \frac{2}{\epsilon} - \log \Delta - \gamma_E + \log(4\pi) \right) - \frac{m^2(1 + z)^2}{\Delta} \right]. \tag{2.a.1}
\]

Let us now compute the one-loop contribution of \( \phi \) to the electron two-point function,

\[
\Sigma_{\phi \phi} \left( \begin{array}{c} p-k \\ \epsilon^- \end{array} \right) \rightarrow \Sigma_{\phi \phi} = \frac{\lambda^2}{2} \int \frac{d^d k}{(2\pi)^d} \frac{i (\ell k + m)}{(p - k)^2 - m^2 + ic}(k^2 - m^2 + ic).
\]

We will define the following variables for Feynman parametrization of the denominator:

\[
\ell \equiv k - z p, \quad \Delta \equiv z(1 - z) \phi^2 + zm^2 \phi + (1 - z)m^2.
\]

We see therefore that

\[
\Sigma_{\phi \phi} = i \frac{\lambda^2}{2} \int \frac{d^d k}{(2\pi)^d} \frac{z \ell \phi + m}{(\ell^2 - \Delta + i\epsilon)^2},
\]

\[
= i \frac{\lambda^2}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(\frac{2 - d}{2})}{\Delta^{-d/2}},
\]

\[
\approx -\frac{\lambda^2}{32 \pi^2} \int \frac{d^d k}{(2\pi)^d} \left[ \frac{z \ell \phi + m}{(\ell^2 - \Delta + i\epsilon)^2} \right],
\]

\[
\therefore \delta Z_2 = \left. \frac{\partial \Sigma_{\phi \phi}}{\partial \phi} \right|_{\phi = m} = -\frac{\lambda^2}{32 \pi^2} \int \frac{d^d k}{(2\pi)^d} \left[ \frac{1}{(\ell^2 - \Delta + i\epsilon)^2} \right],
\]

\[
\therefore \delta Z_2 = -\frac{\lambda^2}{32 \pi^2} \int \frac{d^d k}{(2\pi)^d} \left[ \frac{1}{(\ell^2 - \Delta + i\epsilon)^2} \right]. \tag{2.a.2}
\]
Let us now compute the difference $\delta Z_2 - \delta Z_1$. We see that
\[
\delta Z_2 - \delta Z_1 = \frac{\lambda^2}{32\pi^2} \int_0^1 dz \left[ (1 - 2z) \log \frac{1}{\Delta} + (1 - 2z) \left( \frac{2}{c} - \gamma_E + \log(4\pi) \right) - (1 - z) - \frac{m^2(1 - z)(1 + z)}{\Delta} (2z - (1 + z)) \right],
\]
\[
= \frac{\lambda^2}{32\pi^2} \int_0^1 dz \left[ (1 - 2z) \log \frac{1}{\Delta} - (1 - z) + \frac{m^2(1 - z)^2(1 + z)}{\Delta} \right],
\]
\[
= \frac{\lambda^2}{32\pi^2} \int_0^1 dz \left[ (1 - z) - \frac{m^2(1 - z)(1 - z^2)}{\Delta} - (1 - z) + \frac{m^2(1 - z)^2(1 + z)}{\Delta} \right],
\]
\[
= \frac{\lambda^2}{32\pi^2} \int_0^1 dz \left[ -\frac{m^2(1 - z)^2(1 + z)}{\Delta} + \frac{m^2(1 - z)^2(1 + z)}{\Delta} \right].
\]
\[
\therefore \delta Z_2 - \delta Z_1 = 0.
\]

(2.a.3)

We can expect that $Z_1 = Z_2$ quite generally in this theory because our proof of the Ward-Takahashi identity relied, fundamentally, on the local $U(1)$ gauge invariance of the $A_\mu$ term in the Lagrangian which is not altered by the addition of the scalar $\phi$.

b) Let us now consider the renormalization of the $\bar{\psi}\psi$ vertex in this theory.

The two diagrams at the one-loop level that contribute to $\bar{\pi}(p')\delta\Gamma u(p)$ are

\[
\begin{align*}
&\begin{array}{c}
\bar{\pi}(p')\delta\Gamma u(p) = \int \frac{d^4k}{(2\pi)^4} \bar{\pi}(p') \left[ (i\lambda \sqrt{2}) \left( \frac{i}{(p - k)^2 - m^2 + i\epsilon} - \frac{i(\gamma + m)}{i\epsilon} \right) \right. \\
&\left. \quad \times \frac{(1 + z)^2}{(k + q)^2 - m^2 + i\epsilon} \right] \frac{(1 + z)^2}{(k + q)^2 - m^2 + i\epsilon} \frac{(1 + z)^2}{(k + q)^2 - m^2 + i\epsilon} \left( -i \frac{\lambda}{\sqrt{2}} \right) \right] \Gamma_{\mu}(p) u(p).
\end{array}
\end{align*}
\]

Taking the limit where $q \to 0$ and introducing the variables
\[
\ell \equiv k - z, \quad \Delta_1 \equiv (1 - z)^2 m^2 + z m_\phi^2, \quad \text{and} \quad \Delta_2 \equiv (1 - z)^2 m^2 + z m^2,
\]
this becomes,
\[
\bar{\pi}(p)\delta\Gamma u(p) = \int \frac{1}{dz} \left( \frac{1}{dz} \int \frac{d^4k}{(2\pi)^4} \bar{\pi}(p) \right) \left[ i\lambda^2 \ell^2 (1 + z)^2 m^2 \left( \frac{1}{(\ell^2 - \Delta_1 + i\epsilon)^3} - \frac{2i\epsilon}{(\ell^2 - \Delta_2 + i\epsilon)^3} \right) \right] \frac{d\ell^2}{(\ell^2 - \Delta_2 + i\epsilon)^3} \quad u(p).
\]

Therefore,
\[
\delta Z_1' = -\delta F_1' = \int \frac{1}{dz} \left( \frac{1}{dz} \int \frac{d^4k}{(2\pi)^4} \bar{\pi}(p) \right) \left[ -i\lambda^2 \ell^2 (1 + z)^2 m^2 \left( \frac{1}{(\ell^2 - \Delta_1 + i\epsilon)^3} + \frac{2i\epsilon}{(\ell^2 - \Delta_2 + i\epsilon)^3} \right) \right] \frac{d\ell^2}{(\ell^2 - \Delta_2 + i\epsilon)^3} \quad u(p).
\]

\[
= \int \frac{1}{dz} \left( \frac{1}{dz} \int \frac{d^4k}{(2\pi)^4} \bar{\pi}(p) \right) \left[ -i\lambda^2 \ell^2 (1 + z)^2 m^2 \left( \frac{1}{(\ell^2 - \Delta_1 + i\epsilon)^3} + \frac{2i\epsilon}{(\ell^2 - \Delta_2 + i\epsilon)^3} \right) \right] \frac{d\ell^2}{(\ell^2 - \Delta_2 + i\epsilon)^3} + \text{finite terms},
\]
\[
= \int \frac{1}{dz} \left( \frac{1}{dz} \int \frac{d^4k}{(2\pi)^4} \bar{\pi}(p) \right) \left[ -i\lambda^2 \ell^2 (1 + z)^2 m^2 \left( \frac{1}{(\ell^2 - \Delta_1 + i\epsilon)^3} + \frac{2i\epsilon}{(\ell^2 - \Delta_2 + i\epsilon)^3} \right) \right] \frac{d\ell^2}{(\ell^2 - \Delta_2 + i\epsilon)^3} + \text{finite terms},
\]
\[
= \int \frac{1}{dz} \left( \frac{1}{dz} \int \frac{d^4k}{(2\pi)^4} \bar{\pi}(p) \right) \left[ -i\lambda^2 \ell^2 (1 + z)^2 m^2 \left( \frac{1}{(\ell^2 - \Delta_1 + i\epsilon)^3} + \frac{2i\epsilon}{(\ell^2 - \Delta_2 + i\epsilon)^3} \right) \right] \frac{d\ell^2}{(\ell^2 - \Delta_2 + i\epsilon)^3} + \text{finite terms},
\]
\[
= \int \frac{1}{dz} \left( \frac{1}{dz} \int \frac{d^4k}{(2\pi)^4} \bar{\pi}(p) \right) \left[ -i\lambda^2 \ell^2 (1 + z)^2 m^2 \left( \frac{1}{(\ell^2 - \Delta_1 + i\epsilon)^3} + \frac{2i\epsilon}{(\ell^2 - \Delta_2 + i\epsilon)^3} \right) \right] \frac{d\ell^2}{(\ell^2 - \Delta_2 + i\epsilon)^3} + \text{finite terms},
\]
\[
= \int \frac{1}{dz} \left( \frac{1}{dz} \int \frac{d^4k}{(2\pi)^4} \bar{\pi}(p) \right) \left[ -i\lambda^2 \ell^2 (1 + z)^2 m^2 \left( \frac{1}{(\ell^2 - \Delta_1 + i\epsilon)^3} + \frac{2i\epsilon}{(\ell^2 - \Delta_2 + i\epsilon)^3} \right) \right] \frac{d\ell^2}{(\ell^2 - \Delta_2 + i\epsilon)^3} + \text{finite terms},
\]
\[
\therefore \delta Z_1' = \frac{1}{\epsilon} \left( \frac{\lambda^2}{16\pi^2} \left( 2 - \log \Delta_1 - \gamma_E + \log(4\pi) - \frac{1}{2} \right) - \frac{2\alpha}{\pi} \left( \frac{2}{\epsilon} - \log \Delta_2 - \gamma_E + \log(4\pi) - 1 \right) \right) + \text{finite terms}.
\]

(2.b.2)
Now let us compute $\delta Z'_2$. We see that this factor comes from the diagrams,

\[ e^- \quad p \quad k \quad p \quad + \quad e^- \quad p \quad k \quad p \]

We see that we have already computed both of these contributions; the first diagram’s contribution was computed above and the second diagram’s contribution was computed in homework 6.

Therefore, we note that

\[ \delta Z'_2 = \frac{1}{\epsilon} \left( -\frac{\lambda^2}{32\pi^2} - \frac{\alpha}{2\pi} \right) + \text{finite terms}. \]  

\[ \text{(2.b.3)} \]

Combining these results, we have that

\[ \therefore \delta Z'_2 - \delta Z'_1 = \frac{3}{\epsilon} \left( \frac{\alpha}{2\pi} - \frac{\lambda^2}{32\pi^2} \right) + \text{finite terms} \neq 0. \]  

\[ \text{(2.b.4)} \]