1. a) We are given complex scalar Lagrangian,

\[ \mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi. \]

It is clear that the canonical momenta of the field are

\[ \pi = \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi)} = \partial_0 \phi^*; \]
\[ \pi^* = \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi^*)} = \partial_0 \phi. \]

The canonical commutation relations are then

\[ [\phi(x), \partial_0 \phi^*(y)] = [\phi^*(x), \partial_0 \phi(y)] = i\delta^{(3)}(x - y), \]

with all other combinations commuting. As in Homework 2, the Hamiltonian can be directly computed,

\[ H = \int d^3x \mathcal{H} = \int d^3x (\pi \partial_0 \phi - \mathcal{L}), \]
\[ = \int d^3x (\pi^* \pi - 1/2 \pi^* \pi + 1/2 \nabla \phi^* \nabla \phi + 1/2 m^2 \phi^* \phi), \]
\[ = \frac{1}{2} \int d^3x (\pi^* \pi + \nabla \phi^* \nabla \phi + m^2 \phi^* \phi). \]

We can use this expression for the Hamiltonian to find the Heisenberg equation of motion. We have

\[ i\partial_0 \phi(x) = \left[ \phi(x), \frac{1}{2} \int d^3y (\pi^*(y)\pi(y) + \nabla \phi^*(y) \nabla \phi(y) + m^2 \phi^*(y) \phi(y)) \right], \]
\[ = \frac{1}{2} \int d^3y [\phi(x), \pi(y)] \pi^*(y), \]
\[ = \frac{i}{2} \int d^3y \delta^{(3)}(x - y) \pi^*(y), \]
\[ = \frac{i}{2} \pi^*(x). \]

Analogously, \( i\partial_0 \phi^*(x) = \frac{i}{2} \pi(x) \). Notice that this derivation used the fact that \( \phi \) commutes with everything in \( \mathcal{H} \) except for \( \pi \). Before we compute the commutator of \( \pi^*(x) \) with the Hamiltonian, we should re-write \( \mathcal{H} \) as PS did so that our conclusion will be more lucid. We have from above that

\[ H = \frac{1}{2} \int d^3x (\pi^* \pi + \nabla \phi^* \nabla \phi + m^2 \phi^* \phi). \]

We can evaluate the middle term in \( H \) using Green’s Theorem (essentially integration by parts). We will assume that the surface term vanishes at infinity because the fields must. This allows us to write the Hamiltonian as,

\[ H = \frac{1}{2} \int d^3x (\pi^* \pi + \phi^* (-\nabla^2 + m^2) \phi). \]
Because the field is no longer purely real, we cannot assume that the coefficient of $\phi$.

I computed the conserved Noether charge in Homework 2 as

$$\int d^3 y (-\nabla^2 + m^2) \phi(y),$$

$$\int d^3 y \phi(y) \phi(x),$$

$$\int d^3 y (-\nabla^2 + m^2) \phi(x).$$

Combining the two results, it is clear that

$$\partial_\mu^2 \phi(x) = (\nabla)^2 - m^2) \phi(x),$$

$$\implies (\partial_\mu \nabla^\mu + m^2) \phi = 0.$$

This is just the Klein-Gordon equation. The result is the same for the complex conjugate field.

b) Because the field is no longer purely real, we cannot assume that the coefficient of $e^{i p \cdot x}$ in the ladder-operator Fourier expansion is the adjoint of the coefficient of $e^{-i p \cdot x}$. Therefore we will use the operator $b$. The expansion of the fields are then

$$\phi(x^\mu) = \int \frac{d^3 p}{(2\pi)^3} \sqrt{2\omega_p} \left( a_p e^{-i p \cdot x} + b_p^\dagger e^{i p \cdot x} \right);$$

$$\phi^*(x^\mu) = \int \frac{d^3 q}{(2\pi)^3} \sqrt{2\omega_q} \left( a_q^\dagger e^{i q \cdot x} + b_q e^{-i q \cdot x} \right).$$

It is easy to show that these allow us to define $\pi$ and $\pi^*$ in terms of $a$ and $b$ operators as well. These become,

$$\pi(x^\mu) = -i \partial_\mu \phi^*(x^\mu) = \int \frac{d^3 q}{(2\pi)^3} i \sqrt{\frac{\omega_q}{2}} \left( a_q^\dagger e^{i q \cdot x} - b_q e^{-i q \cdot x} \right);$$

$$\pi^*(x^\mu) = -i \partial_\mu \phi(x^\mu) = \int \frac{d^3 p}{(2\pi)^3} i \sqrt{\frac{\omega_p}{2}} \left( -a_p e^{-i p \cdot x} + b_p^\dagger e^{i p \cdot x} \right).$$

These allow us to directly demonstrate that

$$[\phi(x^\mu), \pi(y^\mu)] = \int d^3 x d^3 y \left[ \left( \frac{\omega_q}{\omega_p} \right) \left( [a_p, a_q^\dagger] - [b_p, b_q] \right) \right] e^{-i (p \cdot x - q \cdot y)},$$

$$= i \delta^{(3)}(x - y),$$

while noting that

$$[a_p, a_q^\dagger] = [b_p, b_q^\dagger] = (2\pi)^3 \delta^{(3)}(p - q),$$

and all other terms commute. This implies that there are in fact two entirely different sets of particles with the same mass: those created by $b^\dagger$ and those created by $a^\dagger$.

c) I computed the conserved Noether charge in Homework 2 as

$$j^\mu = i \left( \phi \partial_\mu \phi^* - \phi^* \partial_\mu \phi \right).$$

We integrate this over all space to see the conserved current in the 0 component. When expressing $\phi$ and $\pi$ in terms of ladder operators, we can evaluate this directly.

$$Q = \frac{i}{2} \int d^3 \phi^{\dagger}(x) \pi\phi(x),$$

$$= \frac{i}{2} \int \frac{d^3 x d^3 y d^3 q}{(2\pi)^6} \left( a_p a_q^\dagger e^{i x \cdot (q - p)} - a_p b_q e^{-i x \cdot (p + q) - b_p^\dagger a_q^\dagger e^{i x \cdot (q - p)} - b_p^\dagger b_q e^{-i x \cdot (q - p)} \right) - c.c.,$$

$$= \frac{i}{2} \int \frac{d^3 p d^3 q}{(2\pi)^3} \left( a_p a_q^\dagger \delta^{(3)}(p - q) - a_p b_q \delta^{(3)}(p + q) + b_p^\dagger a_q^\dagger \delta^{(3)}(p + q) - b_p^\dagger b_q \delta^{(3)}(p - q) \right) - c.c.,$$

$$= \frac{i}{2} \int \frac{d^3 p}{(2\pi)^3} \left( a_p a_q^\dagger - a_p b_{-p} + b_p^\dagger a_{-q}^\dagger - b_p^\dagger b_{-q} \right) - c.c.,$$

$$= \frac{i}{2} \int \frac{d^3 p}{(2\pi)^3} \left( a_p a_q^\dagger - b_p^\dagger b_{-p} \right).$$
2. a) We are asked to compute the general, K-type Bessel function solution of the Wightman propagator,

\[ D_W(x) \equiv \langle 0 | \phi(x) \phi(0) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ipx}. \]

Because \( x \) is a space-like vector, there exists a reference frame such that \( x^0 = 0 \). This implies that \( x^2 = -x^2 \). And this implies that \( px = -p \cdot x = -|p||x|\cos(\theta) = -|p|\sqrt{-x^2}\cos(\theta) \). We can then write \( D_W(x) \) in polar coordinates as

\[
D_W(x) = \frac{1}{(2\pi)^3} \int_0^{2\pi} d\phi \int_0^\pi e^{i|p|\sqrt{-x^2} \cos(\theta)} \int_0^{\infty} p^2 dp \frac{1}{2\sqrt{p^2 + m^2}}.
\]

\[
= \frac{1}{(2\pi)^3} \int_0^\pi d\theta \ e^{i|p|\sqrt{-x^2} \cos(\theta)} \int_0^{\infty} p^2 dp \frac{1}{2\sqrt{p^2 + m^2}},
\]

(\text{where } \xi = \cos(\theta))

\[
= \frac{1}{4\pi^2} \int_0^{\infty} p^2 dp \frac{1}{\sqrt{p^2 + m^2}} \frac{1}{|p|\sqrt{-x^2}} \left( e^{i|p|\sqrt{-x^2}} - e^{-i|p|\sqrt{-x^2}} \right).
\]

Gradsteyn and Ryzhik’s equation (3.754.2) states that for a K Bessel function,

\[
\int_0^{\infty} dx \frac{\cos(ax)}{\sqrt{\beta^2 + x^2}} = K_0(a\beta).
\]

By differentiating both sides with respect to \( a \), it is shown that

\[
- \int_0^{\infty} dx \frac{a \sin(ax)}{\sqrt{\beta^2 + x^2}} = -\beta K_0'(a\beta) = \beta K_1(a\beta).
\]

We can use this identity to write a more concise equation for \( D_W(x) \). We may conclude

\[
D_W(x) = \frac{m}{4\pi^2 \sqrt{-x^2}} K_1(m \sqrt{-x^2}).
\]

b) We may compute directly,

\[
iD(x) = \langle 0 | [\phi(x), \phi(0)] | 0 \rangle,
\]

\[
= \langle 0 | \phi(x) \phi(0) | 0 \rangle - \langle 0 | \phi(0) \phi(x) | 0 \rangle,
\]

\[
= D_W(x) - D_W(-x),
\]

\[ \implies D(x) = i(D_W(-x) - D_W(x)). \]

Similarly,

\[ D_1(x) = \langle 0 | [\phi(x), \phi(0)] | 0 \rangle = D_W(x) + D_W(-x). \]

It is clear that both function ‘die off’ very rapidly at large distances. I was not able to conclude that they were truly vanishing, but they are certainly nearly-so at even moderately small distances.