Substituting our expressions for \( \alpha \) and \( \beta \) and setting the above to zero, we have

\[
0 = \left( 1 - \frac{n_2}{n_1} \right) \alpha + \left( 1 + \frac{n_2}{n_1} \right) \beta, \\
= \left( 1 - \frac{n_2}{n_1} \right) \left( 1 + \frac{1}{n_2} \right) e^{i\sigma(k_2-k_1)} + \left( 1 + \frac{n_2}{n_1} \right) \left( 1 - \frac{1}{n_2} \right) e^{i\sigma(k_2+k_1)}, \\
= \left( 1 - \frac{n_2}{n_1} \right) \left( 1 + \frac{1}{n_2} \right) e^{-i\sigma k_2} + \left( 1 + \frac{n_2}{n_1} \right) \left( 1 - \frac{1}{n_2} \right) e^{i\sigma k_2}, \\
= \left( 1 + \frac{1}{n_2} - \frac{n_2}{n_1} \right) e^{i\sigma k_2} + \left( 1 - \frac{1}{n_2} + \frac{n_2}{n_1} \right) e^{i\sigma k_2}, \\
= \frac{n_1 - 1}{n_2} \cos(\sigma k_2) + \frac{n_2 - n_1}{n_1 n_2} i \sin(\sigma k_2) = 0.
\]

For this to vanish, each of the two contributions must vanish separately. Because \( n_1 - 1 \geq 0 \) and \( \sin(\sigma k_2) \) can be nonzero in general, this implies that \( \cos(\sigma k_2) = 0 \) and that \( \frac{n_1}{n_2} - 1 = 0 \). The cosine vanishes if and only if \( \sigma = \frac{\pi(i + 1/2)}{n_2} \) for some \( i \in \mathbb{Z} \). Therefore, we have that

\[
\therefore n_2 = \sqrt{n_1} \quad \text{and} \quad \sigma = \frac{\pi(i + 1/2) \epsilon c}{n_2 \omega} \quad | \quad i \in \mathbb{Z}.
\]

**Problem 7.3**

Let us consider two parallel semi-infinite slabs of uniform, isotropic, nonpermeable, lossless dielectrics with index \( n \) that are separated by an air gap \( (n = 1) \) of width \( d \), and a linearly polarized electromagnetic plane wave incident on the slabs with an angle \( \phi \) to the normal of the air gap with its polarization perpendicular to the plane of incidence.

**a)** We are to calculate the transmission and reflection from the gap.

We will assess this situation with the same notation as problem 7.2 above where we have the various electric fields given by

\[
E_1 = a e^{i k_1 x} + b e^{-i k_1 x}, \\
E_2 = a e^{i k_2 x} + b e^{-i k_2 x}, \\
E_3 = n_2 e^{i k_1 x},
\]

where \( k_1 = n\epsilon c/\omega \) and \( \kappa = c/\omega \) and \( a, b, \alpha, \beta, \) and \( \eta \) are constants. Notice that the transmission fraction is given by \( T = |\eta|^2/|a|^2 \), the ratio of the final wave magnitude to the initial, incoming wave magnitude. Also, the reflection fraction is obviously \( R = 1 - T \). Therefore it will be very useful for us to solve for \( a \) in terms of \( \eta \) (or vice versa).

The analysis is similar to that of problem 7.2 except that we must now allow for waves which are not normal to the surface. The conditions are spelled out rather explicitly in Jackson’s section 7.3 and it is obvious that our continuity requirements become the following system of constraints,

\[
a + b = \alpha + \beta \\
a - b = \frac{\cos \phi}{n \cos \phi} (\alpha - \beta)
\]

where \( \lambda \equiv d/\cos \phi \), the distance travelled by the wave in going between the two slabs. For the sake of simplicity, set \( \xi \equiv n \cos \phi/\cos \phi \). Solving for \( \alpha, \beta \) in terms of \( \eta \), we see that

\[
\alpha = \frac{1}{2} (1 + \xi) \eta e^{i\lambda(k_1 - k)}; \\
\beta = \frac{1}{2} (1 - \xi) \eta e^{id(k_1 + k)}.
\]
Solving for \( a \) in terms of \( \alpha, \beta \) in our head and substituting for the above expressions for \( a, \beta \) in terms of \( \eta, \lambda \) we see that in the case where \( \xi, \lambda \) are real,

\[
\begin{align*}
& a = \frac{1}{2} \left[ \left( 1 + \frac{1}{\xi} \right) \alpha + \left( 1 - \frac{1}{\xi} \right) \beta \right], \\
& = \frac{1}{2} \left[ \left( 1 + \frac{1}{\xi} \right) \frac{1}{2} (1 + \xi) \eta e^{i\lambda(k_1-k)} + \left( 1 - \frac{1}{\xi} \right) \frac{1}{2} (1 - \xi) \eta e^{i\lambda(k_1+k)} \right], \\
& = \frac{\eta e^{i\lambda k_1}}{4} \left[ \left( 1 + \frac{1}{\xi} \right) (1 + \xi) e^{-i\lambda k} + \left( 1 - \frac{1}{\xi} \right) (1 - \xi) e^{i\lambda k} \right], \\
\implies |a|^2 &= \frac{\eta^2}{16} \left[ (1 + \frac{1}{\xi})^2 (1 + \xi)^2 + (1 - \frac{1}{\xi})^2 (1 - \xi)^2 - 4 \eta^2 (1 + \xi)(1 - \xi) + 4 \eta^2 \frac{1}{\xi} \right]. \\
\therefore \mathcal{T} &= \left( \frac{|a|^2}{|\eta|^2} \right)^{-1} = \frac{(4\xi)^2}{(\xi + 1)^4 + (\xi - 1)^4 - 2 (1 - \xi^2) \cos(2k\lambda)}. \\
\end{align*}
\]

And it is obvious that

\[
\mathcal{R} = 1 - \mathcal{T}.
\]

The case that \( \xi, \lambda \) are real is exactly that when the angle of incidence is less than the critical angle. In the case where the angle is greater than critical, then \( \xi, \lambda \) are both purely imaginary. This changes some of the algebra above. Specifcally, we have that

\[
\begin{align*}
& a = \frac{\eta e^{i\lambda k_1}}{4} \left[ \left( 1 + \frac{1}{\xi} \right) (1 + \xi) e^{-i\lambda k_1} + \left( 1 - \frac{1}{\xi} \right) (1 - \xi) e^{i\lambda k_1} \right], \\
& \implies |a|^2 = \frac{\eta^2 e^{2i\lambda k_1}}{16|\xi|^2} \left[ (\xi + 1)^4 e^{-2i\lambda k} + (\xi - 1)^4 e^{2i\lambda k} + 2 \Re \left( (\xi - 1)^2 (\xi + 1)^2 \right) \right], \\
& = \frac{\eta^2 e^{2i\lambda k_1}}{16|\xi|^2} \left[ (\xi + 1)^4 e^{-2i\lambda k} + (\xi - 1)^4 e^{2i\lambda k} - 2 (1 - 6|\xi|^2 + |\xi|^4) \right], \\
\therefore \mathcal{T} &= \left( \frac{|a|^2}{|\eta|^2} \right)^{-1} = \frac{(4|\xi| e^{-i\lambda k_1})^2}{(\xi + 1)^4 e^{-2i\lambda k} + (\xi - 1)^4 e^{2i\lambda k} - 2 (1 - 6|\xi|^2 + |\xi|^4)}. \\
\end{align*}
\]

And it is obvious that

\[
\mathcal{R} = 1 - \mathcal{T}.
\]

(b) In the case there \( \varphi \) is greater than the critical angle, we are to discuss the limits of \( d \to 0 \) and \( d \to \infty \).

When \( d \to 0 \), we see that all of the exponentials become the identity, we have that

\[
\mathcal{T} \to \frac{16|\xi|^2}{\xi + 1)^4 + (\xi - 1)^4 - 2 (1 - 6|\xi|^2 + |\xi|^4)} = 1.
\]

Similarly, we see that as \( d \to \infty \) the transmission decays exponentially because 

\[-id/\cos \varphi' = -d/|\cos \varphi'|\]

which is exactly what we expect.